A New Calibration Method for Stock-Flow Consistent Models with an Application for Neo-Kaleckian Growth Model with Endogenous Money Supply ♣ ♠

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Abstract: The objective of the present article is to use the new methodology developed by Costa Santos (2017) in order to simulate a simple SFC Neo-Kaleckian growth model and then compare the results thus obtained with the analytical-numerical method for stability analysis. Regarding the specification of behavior equations of our SFC neo-kaleckian growth model, our approach is based on the work of Lavoie and Godley (2012) for investment demand. After calibrating and setting the initial conditions of the model, we run the baseline simulation in order to analyze the specific properties of the time path of endogenous variables and calculate their steady-state values. The general model dynamics and stability are analyzed after reducing the twenty equations of the previous model to a system of only 3 equations, where the endogenous variables (wealth, all bills issued and corporate debt) are normalized by capital stock. Then we use Monte Carlo simulation with uniform distribution for generate 10^8 set of parameters. Using two distinct methodologies, we identified the steady-state values and whether they generated stable equilibria. The first methodology, M1, is usual for equilibrium analysis in difference equations while the second, M2, is an algorithm developed by Costa Santos (2017) that simplifies the process. Finally, we identify that M2 generates similar results to M1 and provides an escape route for large models that are difficult to solve analytically.

Keywords: Post-Keynesian Growth, Stock-Flow Consistency, Simulation Models

JEL Classification – E12, E37, P10

September 2019

♣ The authors acknowledged the useful comments made by Genaro Zezza. Usual disclaimer applies.

♠ Paper prepared to be delivered at 23 FMM Conference: The Euro at 20 – Macroeconomic Challenges, to be held in Berlin from 24 to 26 of October, 2019.

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1. Introduction

The usual treatment given in the SFC literature for solving the models is to work with a numerical solution for a set of parameters considered in the analysis. In general, the models developed are understood as a linear system $n \times n$ where the current variables are functions of the other variables, current and lagged. According to Caverzasi and Godin's (2014) survey of the current state of literature of post-Keynesian SFC-type models, there are two ways to solve an economic model: numerically or analytically. The authors also point out that there is the possibility of deductively solving a model, but this is not a proper way of solving it.

Solving a model numerically implies dealing with the following fundamental questions: (i) how to determine the value of the parameters and what the initial value of the endogenous variables; (ii) how to use the results of the simulations. The first question can be answered with two methodologies: (a) estimation or (b) calibration. Estimation is statistical / econometric methodology to determine the value of the parameters to be used in the model. Calibration, in turn, consists in the process of determining values for the parameters and initial stocks using stylized facts or practical rules. The problem that arises from the first methodology derives from the implicit premise that the parameters are constant over time, thus opening a door for Lucas (1976) critique, thus making it impossible to analyze the dynamic effects of changes in economic policy.

The problems that arise from the second methodology are related to how to use the results of the simulation. Two approaches have been used in the literature. The first is to let the model start at *steady-state*, giving an exogenous shock to it. The second is to draw a *baseline scenario*, where converge to *steady-state* is not imposed from outside; and from this scenario modify some parameters in order to see the behavior of the model. Our approach to the problem at hand is to use the algorithm developed by Costa Santos (2017) to calibrate an economic model and calculating the steady-state values of endogenous variables.

According to O'Shea and Kinsella (2016), there is a numerical algorithm that facilitates the resolution of such models regardless of their size. The algorithm consists in solving for each period t the system through the *Gauss-Seidel method* and to use as initial guess the values obtained in period $t-1$ for the resolution at period t. In this sense, econometric software such as Eviews and R already present a routine that includes the cited algorithm. In this way, a good part of the theoretical work on SFC models developed until now uses these methods to solve proposed models.

However, calibrating models can be a daunting task, since many parameters used are difficult to obtain through econometrics (for lack of data or difficulty in estimation) or previous work. Thus, one of the major criticisms of SFC models is that such models do not provide us with a *general behavior*, but rather a specific behavior for a set of parameters that was used in their calibration. Authors such as Godin et al. (2012) have already made efforts to develop and provide escape routes for other authors who wish to make parameter estimates from the linear programming method.

The alternative created by Costa Santos (2017) consists in mapping the desired (or plausible) intervals to the endogenous variables of the model and, from these conditions, obtain the parameters that lead the model to those intervals. The analytical method of calibration (which we will call M1) is to find the values for fixed points (steady state values) from a system of difference equations; in the sequence, to calculate the Jacobian matrix in the fixed points and from it the real part of the eigenvalues to define local stability. Finally, through the general criterion of stability, define through the eigenvalues module if we are facing a stable or unstable equilibrium. The alternative method (which we shall call M2), created by Costa Santos (2017), consists of a combination of numerical approaches and the use of *brute force* and *computational ignorance* (BFCI) to demonstrate that there is convergence between the results obtained by methods M1 and M2.

Therefore, the aim of the present article is to use the method developed by Costa Santos (2017) in order to simulate a simple SFC Neo-Kaleckian growth models and then compare the results thus obtained with the analytical-numerical method for stability analysis.

Regarding the specification of behavior equations of our SFC Neo-Kaleckian growth model, our approach rests on the work of Lavoie and Godley (2012) for investment demand. More precisely, investment demand is defined in terms of a desired rate of capital accumulation, as in canonical Kaleckian models of growth and distribution such as the ones as developed by Rowthorn (1981), Dutt (1990) and Lavoie (1995). This specification for investment function allows that, once the capacity effect of investment is taken into account, the equilibrium of the model to be defined by *a constant growth rate of real output*. The level of activity is represented in the model by the variable capacity utilization, which is constant and lower than one over the equilibrium path.

Moreover, in steady-state disposable income, capital stock and private wealth all grow at a same constant rate, so a *balanced growth path* exists. We also show that distribution of wealth between money and bonds is also constant in steady-state growth. Finally, it is also shown that along *balanced growth path*, the economy is *dynamically efficient* and firms had a *hedge financial posture*. This means that financial fragility in the sense of Minsky (1986) is ruled out of the model.

Turning to the comparative dynamics of the model, we had performed some numerical simulations about the dynamic effects of shocks over the time path of endogenous variables. We have tested changes in the autonomous rate of capital accumulation, the propensity to consume out of disposable income, the coefficient of profit distribution, the tax rate, the nominal interest rate and the wage share. One important but expected result is that the qualitative effects of changes in the parameters of investment and consumption functions are very similar. Indeed, an increase in the propensity to consume out of disposable income generated an increase in the level of capacity utilization and an increase in the growth rate of capital stock and private wealth; an increase in the autonomous rate of capital accumulation generated just the same qualitative effects.

Another interesting result is about the old "paradox of thrift". An increase in the marginal propensity to consume (a reduction in the marginal propensity to save) resulted in an increase in the level of capacity utilization, and also an increase in the growth rate of capital stock and private wealth[1].

Regarding the effects of changes in fiscal policy over the dynamic behavior of the economy, the model showed that an increase in the tax rate – in a possible attempt of the government to reduce fiscal deficit and the ratio of public debt to GDP – has also no effect over the time path of the public debt to GDP ratio. This can be due to the fact that being growth rate of disposable income higher than interest rate along the balanced growth path then even if government runs a primary deficit than the ratio of public debt to GDP will be decreasing over time, making "fiscal adjustment" unnecessary.

[1] This result replicates the "paradox of thrift" in the growth and distribution model of Joan Robinson. See Harcourt (2006, p. 29).

Besides that, the model showed a clear *wage-led* regime of accumulation[2], since an increase in the wage share resulted in an increase in the growth rate of both capital stock and private wealth and also an increase in the level of capacity utilization.

Finally, the comparison between calibration methodologies to find the steady-state values show that M1 is able to define stable and unstable equilibria, but in some cases the values found may not have any economic sense (wealth and negative public debt, for example). The methodology M2, in turn, necessarily provides parameters for which the endogenous variables make economic sense due to the applied filter.

The paper is organized in eight sections, including the present introduction. Section two is dedicated to the presentation of the new calibration method of SFC models developed by Costa Santos (2017). In section three we presented the accounting structure and the theoretical assumptions of the SFC version of our Neo-Kaleckian growth and distribution model. In section four we will present the behavior equations of the model, that is, its formal theoretical structure. Section five is dedicated to the calibration of the model and the performing of the basic numerical simulation. In section six we perform the comparative dynamic exercises, evaluating the effects of exogenous shocks in some behavior and policy parameters over the dynamic path of the endogenous variables. In section seven, we present the general model dynamics, the steady state and the stability. In section eight, we do some final remarks.

2. Costa Santos´s Calibration Method for SFC Models

As we had told in the introduction, Costa Santos (2017) had developed an alternative method for calibrating SFC models that consists in mapping the desired (or plausible) intervals to the endogenous variables of the model and, from these conditions, obtain the parameters that lead the model to those intervals. The analytical method of calibration (which we will call M1) is to find the values for fixed points from a system of difference equations; in the sequence, to calculate the Jacobian matrix in the fixed points and from it the value of the eigenvalues. Finally, through the general criterion of stability, define through the eigenvalues module if we are facing a stable or unstable equilibrium. If it is a two-dimensional model, we can infer from the trace and the determinant whether

[2] This is not a surprising result since we are supposing the existence of a strong accelerator effect in the investment function. If investment spending was sensitive to changes in the profit share, as in Bhaduri and Marglin (1990), then this result could be reversed to a *profit-led* regime.

it is a stable, unstable fixed point, saddle point, unstable focus, and stable focus. The alternative method (which we shall call M2), created by Costa Santos (2017), consists of a combination of numerical approaches and the use of brute force and computational ignorance (BFCI) to demonstrate that there is convergence between the results obtained by methods M1 and M2.

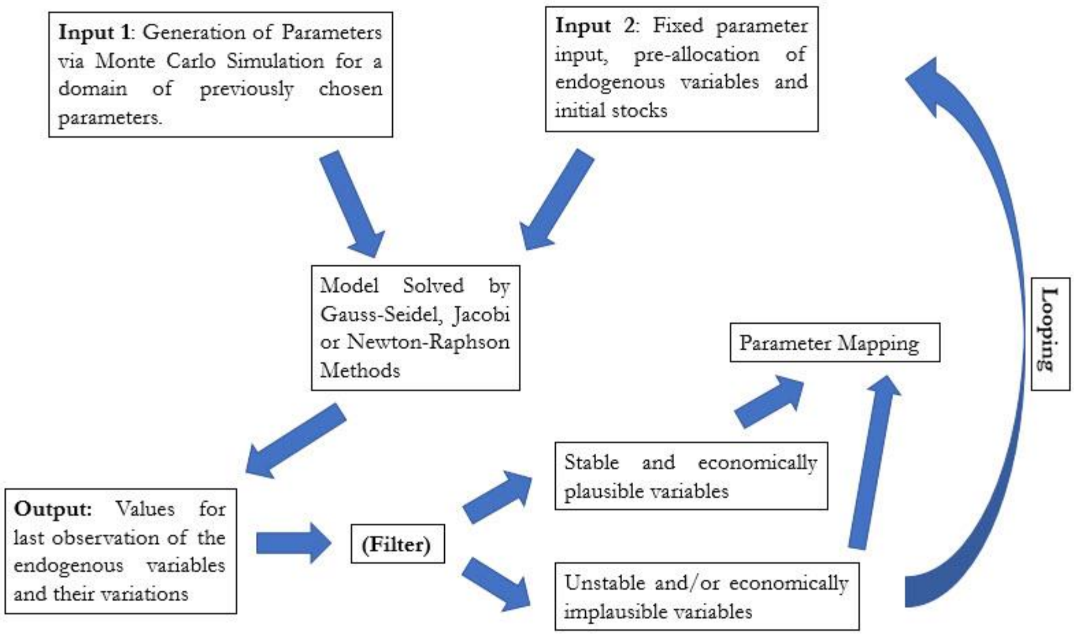
Costa Santos´s method can be described as follows. First, a domain is defined for the set of parameters to be evaluated. We then use the *Monte Carlo simulation* method to generate parameters randomly and uniformly distributed within that domain interval.

For each simulation, m, a set z of model parameters is generated. This is given by: $\{a_i\}^m$, $i = 1,2, \ldots, z$. From this set, the *Gauss-Seidel method* is used to solve the system of equations for each period t, where $t = [1,2, \ldots, n]$. When $t = n$, we stop the algorithm by providing the final set of endogenous variables $\{y^i_{t=n}\}$, $i = 1,2, \ldots, x$. Note that x must be $x \leq z$ and the equations must be linearly independent. From the final endogenous variables, we can obtain the level values and their variations, Δy^i_t, $i = 1,2, \ldots, x$.

Using the criterion that in *steady state* we have that $\Delta y^i_t = 0 \; \forall \; i \in \{1,2, \ldots, x\}$, we have created a filter to select only the parameters for which the condition of $\Delta y^i_t = 0$ was reached. Note that by selecting ex-ante a value n for the final period of calculated t, we may have endogenous variables that are in the convergence path but do not yet have a value equal to zero. Thus, an alternative way that the filter can be flexibilized is to use $\Delta y^i_t < tolerance$, being $tolerance \approx 0$.

Figure 1 below outlines the steps of the algorithm. As previously stated, first a random set of parameters is generated within a previously defined domain using uniform distribution. The model is then solved for all period t. Finally, we obtain the value of the variables and their variations. Filter those variables whose variances are equal to zero (or approximately). By storing the filtered values in an array, we return to step one with new simulations for the parameters. The algorithm is run extensively so that the *computational brute force* provides us with a suitable mapping for the stable parameters.

Figure 1: Schematization of the alternative stability algorithm



3. Accounting Structure and Theoretical Assumptions

We will consider a closed economy (there is no import and export of goods and services and no capital flows) with four sectors: Households, Firms, Government and Central Bank. The balance sheet of these sectors is summarized in Table 1 below.

Table 1: Balance Sheet of the Model

	Households	Firms	Government	Central Bank	∑
Fixed Capital		$+K_f$			$+K_f$
Money	$+H_h$			$-H$	
Bills / Corporate Notes	$+B_h$	$-B_f$	$-B_g$	$+B_{cb}$	
Balance (net worth)	$-V_h$	$-V_f$	$+V_g$		$-K_f$
∑	0	0	0	0	0

Note: Positive variables are assets, while negative ones are liabilities.

Looking at balance sheet, you should notice that the only asset owned by Firms is the fixed capital (tangible goods). Thus all their funds are used to finance the purchase of new fixed capital equipment. We don't consider commercial banks in the composition of monetary system. However, we consider the issuance of corporate notes. We will suppose that corporate notes are *perfect substitutes* of government bills. All funds used to finance firms come from retained profits plus the new corporate bonds issued. Households accumulate financial wealth, which can be allocated in the form of money or buying bills issued by the government or corporate notes issued by firms. The Central Bank is considered as an institution in its own right. The central bank purchases bills from government and also corporate notes from firms, thereby adding to its stock of assets. On its liability side, the central bank provides money to households. This money can take the form of either cash or deposits at the central bank. It is assumed that central bank has zero net worth. The value of bills insured by government is the public debt. As usual all rows and columns must sum zero. The exception is the fixed capital row.

Table 2 shows the transactions-flow matrix of our model. Once again, all columns and all rows must sum zero to ensure that all transactions are taken into account. Thus we avoid *black holes* in the system. The government pays interest arising from government debt both to households and central bank. Interest payments each period are generated by stocks of assets in existence at the end of previous period. Because of this time lag, the rate of interest on bills relevant in period *t* is the rate of interest that was set at the end of previous period, at time t_{-1}.

Table 2: Transactions-flow matrix of Model

		Firms			Central Bank		
	Households	Current	Capital	Government	Current	Capital	∑
Consumption	$-C$	$+C$					0
Government Expenditures		$+G$		$-G$			0
Investment		$+I$	$-I$				0
Wages	$+W$	$-W$					0
Taxes	$-T$			$+T$			0
Interest Payments	$+r_{t-1} * B_{h_{t-1}}$	$-r_{t-1} * B_{f_{t-1}}$		$-r_{t-1} * B_{g_{t-1}}$	$+r_{t-1} * B_{bc_{t-1}}$		0
Central Bank Profits				$+r_{t-1} * B_{bc_{t-1}}$	$-r_{t-1} * B_{bc_{t-1}}$		0
Firms Profits	$+P_h$	$-P$	$+P_f$				0
Change in Money	$-\Delta H$					$+\Delta H$	0
Change in Bills / Corp. Notes	$-\Delta B_h$		$+\Delta B_f$	$+\Delta B_g$		$-\Delta B_{bc}$	0
∑	0	0	0	0	0	0	0

Note: Positive figures denote sources of funds, while negative ones denote uses of fund.

The government savings is the difference between government revenues and expenditures. The increasing of public debt is financed by issuing of new bills. In the opposite way, the fiscal surplus should be used to decrease public debt. By the way, the column sum of government must be zero.

We have set the central bank's net worth in zero, which implies that any profit it makes is always distributed to government. Here, the central bank certainly does make profits since it owns bills which yield interest payments, whereas its liabilities (money) pay no interest.

The household income is given by wages payments by firms, interest payments from government bills and corporate notes and profits distributed by firms. With all these revenues, households pay taxes, purchases goods and services from firms and buys government bills and corporate notes.

Goods sales are the only source of revenues to firms. Households buy a quantity C and government buy a quantity G of goods and services. The current account represents the income flows within the sector, while the capital account represents sources to finance firms. For the model be consistent, the column sum must be zero. The entire resource flows entering should be spent. The firms spend their resources paying wages to households. The difference between all flows constitutes the profit. It is assumed that a share of profits is retained to finance new investments, while the other part is distributed to the households.

4. Model Behavioral Structure

Social account matrix is no able for forecast, by itself, the path taken by the economy. For this purpose, this section will present the behavior equations that explain decision making by economic agents. The behavior of firms, households, government and central bank will be displayed. Decision making is thus represented by aggregate behavior equations (like the consumption and investment function) instead of *Euller* equations coming from some problem of utility maximization. This means that rationality in the model to be presented rationality is better represented by the concept of *procedural rationality* in the sense of Simon (1982)[3]. It is also shown the calculation of short-term output and portfolio decisions. Following the logic of an SFC approach, the model follows the proposal of not generating black holes. Everything that comes from a place is going somewhere else.

[3] For a discussion of alternative concepts of rationality see Possas (1995).

4.1. Firms

Equation (1) defines the level of investment in the current period. The firm must choose the desired level of investment, defining a rate of growth for the capital stock. In the desired rate of capital accumulation rate γ_0 is the parameter that represents the *animal spirits* of entrepreneurs, γ_1 represent the sensitivity of capital accumulation to the level of capacity utilization and the last term γ_2 is the sensitivity of capital accumulation in relation to changes in interest rate. This is a typical Neo-Kaleckian specification for investment function, and it is based on Lavoie and Godley (2012). The actual capacity utilization level of output is given by equations (2). Equation (3) shows that retained profits are equal to *(1-d)* times total profits, where *d* is the coefficient of profit distribution. If retained profits are not enough to finance the desired level of investment, than firms will issue corporate notes in order to get the necessary funds in capital markets, as presented in equation (5). Furthermore, we will suppose that there is no limit for the level of indebtedness of firms, which implies that firms are always capable to secure the required amount of funds to finance the desired level of investment. This means that there is *no financial constraint to investment*. Equation (4) shows the dividends paid to households. Finally, equation (6) shows that the current capital is the sum of the capital inherited plus the current investment.

$$I = \Delta K = (\gamma_0 + \gamma_1 . u - \gamma_2 . r) . K_{-1} \quad (1)$$

$$u = \frac{Y}{K_{-1}} \quad (2)$$

$$P_f = (1 - d) . (Y - W - r_{-1} . B_{f_{-1}}) \quad (3)$$

$$P_h = d . (Y - W - r_{-1} . B_{f_{-1}}) \quad (4)$$

$$B_f = B_{f_{-1}} + I - P_f \quad (5)$$

$$K = K_{-1} + I \quad (6)$$

4.2. Households

The households receive income form three sources. First, as stated on equations (4), all residual profits (difference between retained profit and full profit) are distributed to households. Second, as a payment for labor services, they receive wages, as can be noticed in equation (7). The last source is the interest received for holding government bills and corporate notes. This still could be noticed looking at equation (10). Households pay taxes over total income. For the sake of simplicity, we will assume that government sets a tax rate of θ over wages, interest and dividends. Retained earnings at firms are not taxed. Consumption expenditures, as presented in equation (9), depend both on the stock of wealth and disposable income as in Godley and Lavoie (2007, p. 107). Finally, households´ wealth is defined in period t by the stock accumulated in $t-1$ plus the savings (difference between disposable income and consumption) in t, as presented in equation (8).

$$W = w.Y \tag{7}$$

$$Vh = Vh_{-1} + (YD - C) \tag{8}$$

$$C = \alpha_1.YD + \alpha_2.Vh_{-1} \tag{9}$$

$$YD = (1-\theta).(W + r_{-1}.B_{h-1} + P_h) \tag{10}$$

4.3. Portfolio Decisions

The next equations (11), (12) and (13) defines the behavior of household's portfolio. Equation (11) shows that the demand for money is a portion of the wealth defined by δ_1. Bills and corporate notes are the residual part of wealth that is not allocated in money. δ_1 varies according to the interest rate and preference for liquidity, given by λ_1.

$$H_h = \delta_1.Vh \tag{11}$$

$$B_h = (1-\delta_1).Vh \tag{12}$$

$$\delta_1 = \lambda_1 - r \tag{13}$$

4.4. Government and Central Bank

Now we turn attention to the behavior of government and central bank. Regarding government expenditures, equation (14) states that government set the level of public expenditures in order to obtain a desired ratio to the level of capital stock. This means that government expenditures are driven by capital accumulation. Equation (15) shows that all taxes revenues come from disposable income from households. The implicit idea is that only households are taxed, in other words, retained profits are tax free. Equation (16) states that government issues new bills in order to finance any budget deficit. Households will demand a share of those bills and the corporate notes. The difference between the supply of new bills plus new corporate notes and demand by households will be bought by central bank, that acts as a residual buyer as show equation (18). Note in equation (16) that central bank also buys corporate notes. Finally, in equation (19) we notice that interest rate is an exogenous variable, determined by central bank. The last equation (20) is an identity. It states that all bills insured in the economy are the sum of government bills and corporate notes.

$$G = \gamma . K_{-1} \tag{14}$$

$$T = \theta . (W + r_{-1} . B_{h-1} + P_h) \tag{15}$$

$$\Delta B_g = B_g - B_{g-1} = (G + r_{-1} . B_{s-1}) - (T + r_{-1} . B_{cb-1}) \tag{16}$$

$$\Delta H_s = H_s - H_{s-1} = \Delta B_{cb} \tag{17}$$

$$B_{cb} = B_g + B_f - B_h \tag{18}$$

$$r = \bar{r} \tag{19}$$

$$B_s = B_f + B_g \tag{20}$$

5. Calibration and Baseline Simulation

The model was simulated in *MATLAB 2013 software environment*. We calibrate our model in order to make it as close as possible to what we find in the literature[4]. We have on the table 3 below the values used and the parameter description.

Table 3: Calibration and initial conditions of the Model

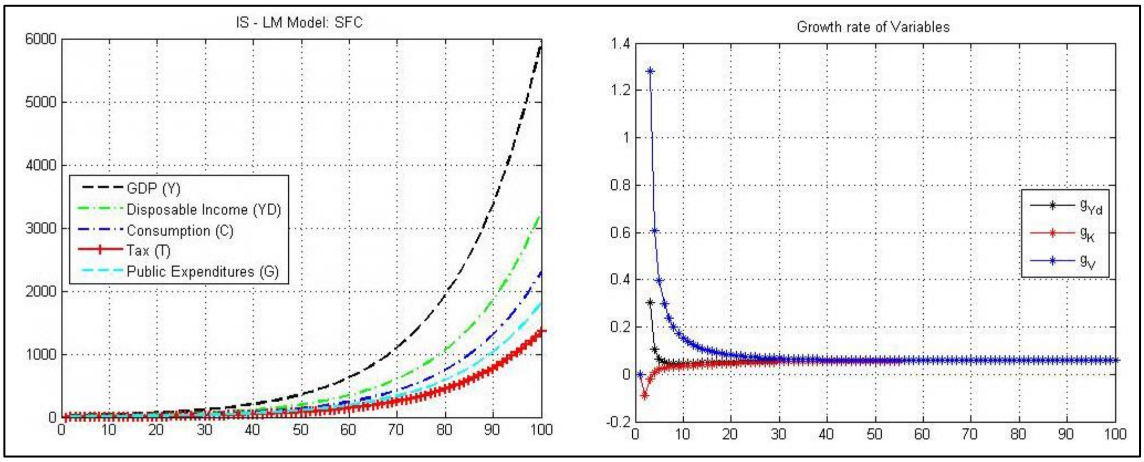
α_1	0.60
α_2	0.02
γ	0.15
γ_0	0.02
γ_1	0.20
γ_2	0.20
θ	0.30
r	0.05
d	0.25
γ	0.15
w	0.60
λ_1	0.20
$K_{Initial}$	100
r	0.05

In the figure 2, we have two quadrants. The first quadrant (west) shows the path of the main aggregates of the real economy. In the second quadrant (east) we have the time path of growth rates of disposable income, capital and wealth. Since all growth rates

[4] For a detailed description of calibration methodology see Oreiro and Ono (2007).

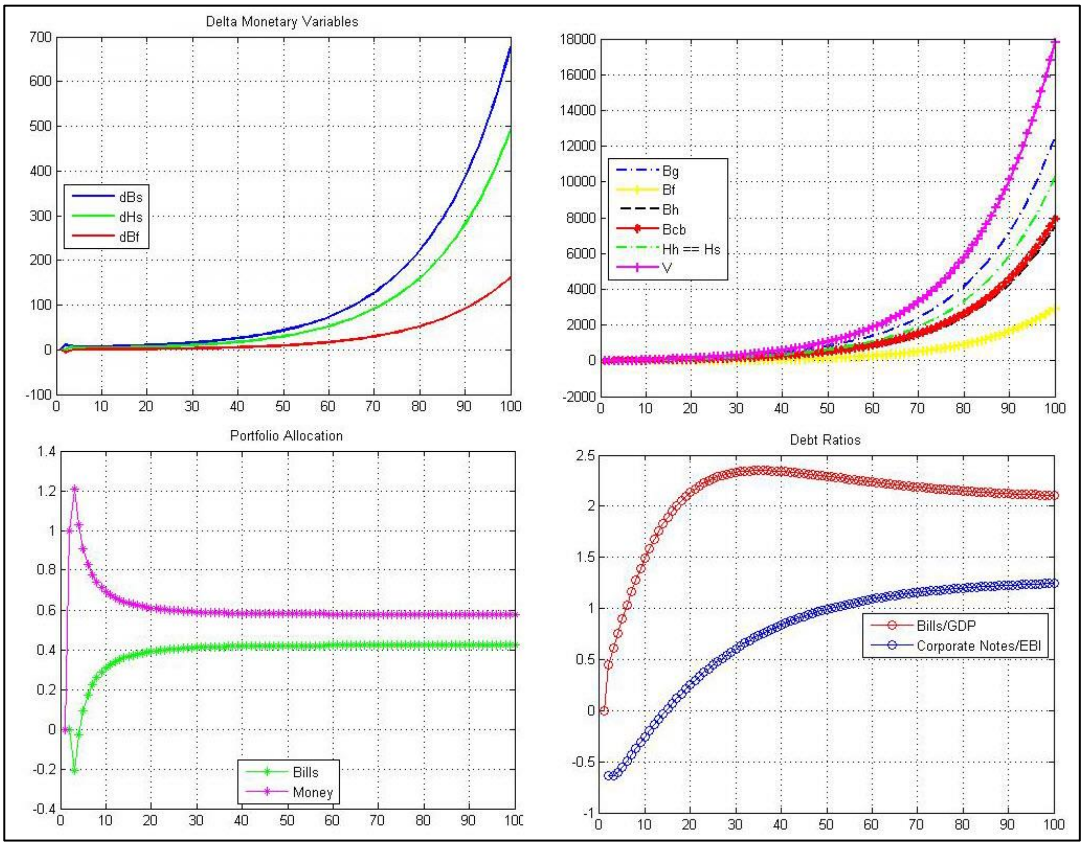
converge to the same positive constant, than we can conclude that the model has a *balanced growth path*.

Figure 2: Main results of the simulated model.



In the figure 3, we have four quadrants. The first quadrant (top/west) shows the paths of monetary variables. In the second quadrant (top/east) we have the behavior of the stocks of financial wealth. The third quadrant (bottom/west) shows the behavior of portfolio composition. As can be seen, after a certain time portfolio composition is kept constant. The fourth quadrant (bottom/east) shows the path of the ratios: Bills/GDP and Corporate Notes/EBI (Earnings before interest). An important point is that the public debt over GDP and private debt over EBI converges to a positive constant rather than keep growing indefinitely. This is an important point to be highlighted because it shows that the model doesn't have an explosive behavior in the public and private debt levels.

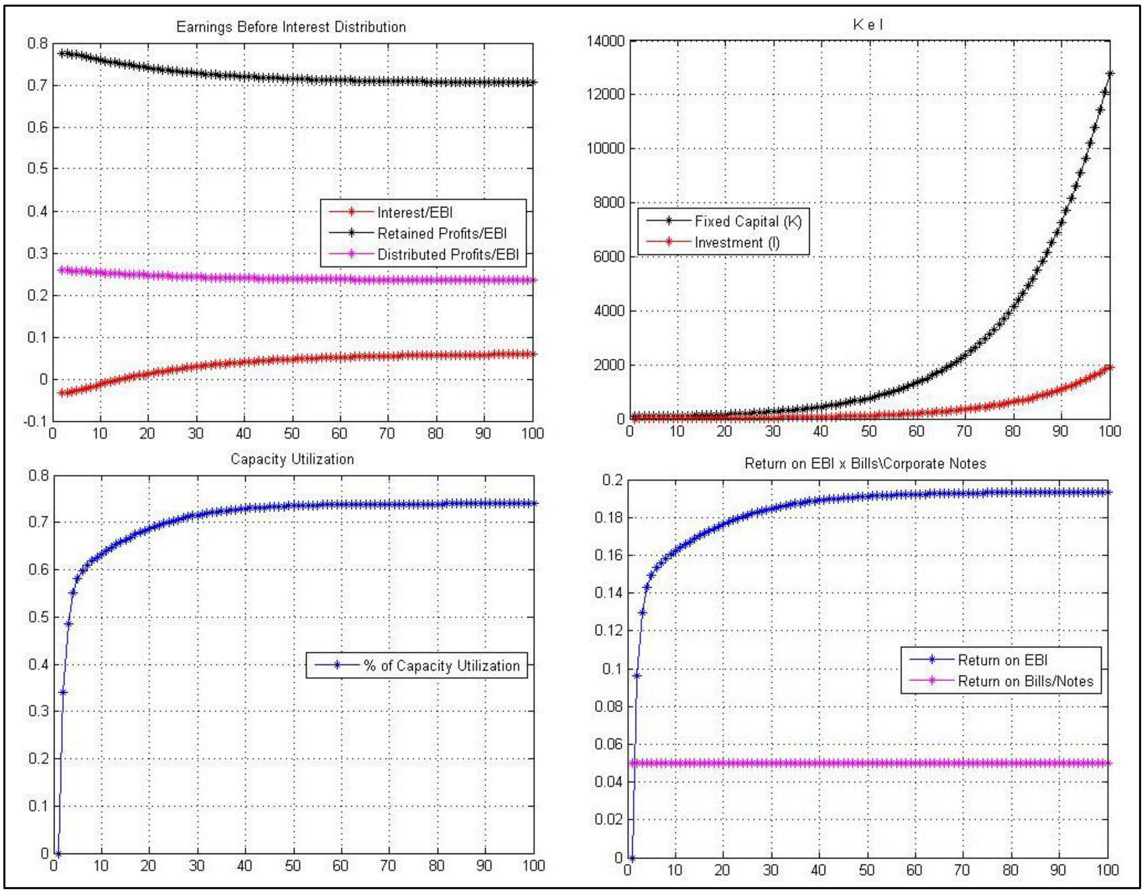
Figure 3: More results of simulated model.



In figure 4, we have in the first quadrant (top/west) the path of the following ratios: Interest/EBI, Retained Profits/EBI and Distributed Profits/EBI. The basic idea in this chart is show that the interest payment doesn't crush the profit nor investment activity. On the second quadrant (top/east) we have the path of fixed capital and investment confirming what was shown in the previous graph. On the third (bottom/west), we have the path of the capacity utilization. As can be seen, capacity utilization converges to a value lower than one, meaning that the model reproduces the traditional Keynesian result of *underemployment equilibrium*[5]. The last quadrant (bottom/east) shows the path of interest rate and of profit rate, that is, the return of the financial investment (bills and corporate notes) and the return on investment in the real economy (fixed capital invested by firms). As we can see, profit rate is consistently higher than interest rate.

[5] Considering the level of capacity utilization as a proxy for employment level.

Figure 4: Firms and Government results of the simulated model.



The data obtained in the *steady state* was summarized and can be seen in Table 4. We have that the GDP growth rate, the fixed capital growth rate and wealth growth rate converge to the same value (5.79% p.p). The return on productive capital remains above the return on financial capital and capacity utilization remains below the rate of full capacity utilization (73.95% of full capacity). The wealth/GDP ratio converges to 2.98, Bills/GDP ratio to 2.065 and Corporate Notes/EBI to 1.281.

Table 4: Main values in Steady-State

Simulation Results on Steady-State	
g_GDP	5.79%
g_K	5.79%
g_V	5.79%
r_K	19.36%
r	5.00%
u	73.95%
V/GDP	2.98
Bills/GDP	2.065
Corporate Notes/EBI	1.281

In Table 4, we can see that along the balanced growth path we have: $r_k > g_k > r$. This means that this economy is *dynamically efficient* (Blanchard and Fischer, 1989, pp.103-4) and firms had a *hedge financial posture* (Foley, 2003, pp.160-01). This last property of the balanced growth means that there is no financial fragility in the long-run equilibrium.

6. Comparative Dynamics

Here, we present the main effects over macroeconomic variables after some shocks. In table 5, we present these results. The shocks were given as follows: it was chosen deliver a shock at the time the model had reached its steady state, in other words, the time which the rates have converged to grow at the same value. Thus, Table 5 shows the observed values of the key variables in the period 1000.

Table 5: Results over the main variables, after shocks.

Shock	g_Y_d	g_V	g_K	Bills/GDP	Notes/EBI	Capacity Utilization	Ret_EBI
Alpha1 +10%	6,34%	6,34%	6,34%	1,58	1,04	76,71%	19,88%
Alpha1 -10%	5,33%	5,33%	5,33%	2,60	1,38	71,67%	18,95%
d +10%	5,34%	5,34%	5,34%	3,32	-0,66	71,69%	17,78%
d -10%	6,27%	6,27%	6,27%	1,14	2,48	76,34%	20,76%
Ghama_0 +10%	6,04%	6,04%	6,04%	1,89	1,36	74,18%	19,48%
Ghama_0 -10%	5,55%	5,55%	5,55%	2,27	1,03	73,74%	19,24%
r +10%	5,78%	5,78%	5,78%	2,12	1,12	74,38%	19,50%
r -10%	5,81%	5,81%	5,81%	2,02	1,28	73,54%	19,23%
Theta +10%	5,31%	5,31%	5,31%	1,95	1,39	71,54%	18,93%
Theta -10%	6,30%	6,30%	6,30%	2,14	1,05	76,50%	19,84%
w +10%	6,52%	6,52%	6,52%	0,75	3,40	77,59%	18,67%
w - 10%	5,12%	5,12%	5,12%	4,10	-1,67	70,64%	20,21%

The effects of a shock in α_1, the propensity to consume from the disposable income: The first row shows the results of +10% in α_1 and in the second row shows a shock of -10% in α_1. The positive shock led to an increase in growth rates, an increase in capacity utilization and a fall in all debt indicators (private and public). The negative shock resulted in a fall of growth rates, a drop in capacity utilization and a rise in all debt indicators.

The effects of a shock in *d*, the coefficient of profit distribution: The third row shows the results of a shock of +10% in *d* and in the fourth row shows a shock of -10% in *d*. The positive shock led to a fall in growth rates, a drop in capacity utilization and a rise in the public debt / GDP and a fall in Notes/EBI. The negative shock resulted in a rise in growth rates, an increase in capacity utilization, a fall in the ratio public debt / GDP and an increase in Notes/EBI. An interesting result can be observed in the dynamics of corporate debt. When we increase *d* by 10%, we are assuming that more profits are distributed to families and less is used to finance business investment. However, as the investment demand doesn't fall, finance is supplied by the issuance of new corporate notes which then increase the firm's debt level. Otherwise, given a shock of -10% in *d*, we have fewer notes to be issued or at worst, may even be repurchased reducing the level of indebtedness of firms.

The effects of a shock in γ_0, the "animal spirits" coefficient: The fifth row shows the results of a shock of +10% in γ_0 and the sixth row shows a shock of -10% in γ_0. The positive shock led to an increase in growth rates, an increase in capacity utilization and a fall in all debt indicators (private and public). The negative shock resulted in a fall of growth rates, a drop in capacity utilization and a rise in all debt indicators.

The effects of a shock in r, the interest rate: The seventh row shows the results of a shock of +10% in r and the eighth row shows a shock of -10% in r. The positive shock led to a fall in growth rates, a decrease in capacity utilization and a rise in all debt indicators (private and public). The negative shock resulted in a rise of growth rates, an increase in capacity utilization and a fall in all debt indicators.

The effects of a shock in θ, the tax rate: The ninth row shows the results of a shock of +10% in θ and the sixth row shows a shock of -10% in θ. The positive shock led to a fall in growth rates, a decrease in capacity utilization and a rise in private debt and a fall in public debt. The negative shock resulted in a rise of growth rates, an increase in capacity utilization and a fall in private debt and a rise in public debt.

The tenth and the eleventh rows shows that in the economy at hand prevails a *wage-led accumulation regime*, since an increase/decrease in wage share is followed by an increase/decrease in the growth rates of capital stock, disposable income and wealth, as well as an increase/decrease in the level of capacity utilization.

7. General Model Dynamics, the Steady State and Stability: a comparison of the two methods

The general dynamics of the system and its stability properties can be analyzed if we reduce the twenty previously equations presented into a system of few equations normalized by the capital. In this section we will show how it is possible to reduce all the equations in a system that depends on the capacity utilization (curve u), the normalized wealth, the normalized debt dynamics (private and public) and the investment equation.

7.1. The Normalized System Dynamics:

7.1.1. The capacity utilization $(\boldsymbol{u})$ curve:

The curve u is formed by the elements of the aggregate demand, being: $u = c + i + g$. Where lowercase letters represent the variables normalized by capital. Using the equations (1), (9), (14) in (2) and after some algebraic manipulation, we have:

$$u = \psi_1.(\rho + \gamma_0 - \gamma_2 r) + \psi_1.\alpha_2.vh_{-1} \tag{S1}$$

Where: $\psi_1 = \frac{1}{1-\alpha_1.(1-\theta).w-\gamma_1}$, which is the Keynesian multiplier.

7.1.2. The Wealth dynamics:

Using equations (4), (8), (9), (10) and (11) and in the sequence organizing the terms and dividing by the inherited capital, we arrive at the new equation that gives the normalized wealth dynamics.

$$vh = \frac{[\kappa_1+\kappa_2.r].vh_{-1}+\kappa_3.u-\kappa_4.r.b_{f_{-1}}}{(1+g)} \tag{S2}$$

Where we made the following parameter substitutions to simplify the model:

$\kappa_1 = 1 - \alpha_2;\ \kappa_2 = (1-\alpha_1).(1-\theta)(1-\delta_1);\ \kappa_3 = (1-\alpha_1).(1-\theta).[w + d.(1 - w)];\ \kappa_4 = (1-\alpha_1).(1-\theta).d$

7.1.3. The Debt Dynamics:

7.1.3.1. All Bills issued (government bills plus corporate notes):

The first debt equation, presented in (S3), is formed by the total bonds issued in the economy. We will start from the premise that firm and government bonds are perfect substitutes[6]. Using the equation (4), (5), (7), (12), (13), (14), (15), (16), (18) and (20) after some algebraic manipulation and normalizing over the capital, we can find (S3).

$$b_s = \frac{b_{s-1}-\sigma_1.u+\sigma_2.r.vh_{-1}-\gamma_2.r+\sigma_3.r.b_{f_{-1}}}{(1+g)} \tag{S3}$$

[6] This hypothesis brings the convenience of creating a model where the bank sector is not needed to finance and allows us to make consistency with the original IS-LM that there is only one interest rate.

Where:

$\sigma_0 = \rho + \gamma_0$; $\sigma_1 = [\theta.w + \theta.d.(1-w) + (1-d).(1-w) + \gamma_1]$; $\sigma_2 = (1 - \theta).(1-\delta_1)$; $\sigma_3 = (1+\theta).d$

7.1.3.2. Corporate Notes:

The (S4) equation presents the dynamics of normalized firm debt.

$$b_f = \frac{b_{f_{-1}} + (1-d).r.b_{f_{-1}} + \gamma_0 - \gamma_2 r + \psi_2.u}{(1+g)} \qquad \text{(S4)}$$

The last equation of the model for the dynamics of the system had been presented before. It is the investment equation, which here will be our fifth and last equation.

$$g = \gamma_0 + \gamma_1.u - \gamma_2 r \qquad \text{(S5)}$$

The short-term relationships between the variables present in the model dynamics can be summarized in the table 3 below. Capacity utilization is only positively related to inherited wealth. Investment is positively related to capacity utilization. Normalized wealth is negatively related to investment and to normalized corporate debt. Depending on the set of parameters can assume a positive (or negative) relationship with capacity utilization and inherited wealth.

Table 3: Short-run links among model variables.

Short-run links among model variables

	u	g	vh_{-1}	b_{s-1}	b_{f-1}
u		-	> 0	-	-
g	> 0		-	-	-
vh	?	< 0	?	-	< 0
b_s	?	< 0	?	> 0	> 0
b_f	?	< 0	< 0		> 0

The system of 5 equations above can be reduced into a system of three equations. To do so, we will introduce the equations (S1) and (S2) into (S2), (S3) and (S4). After some algebraic manipulations and simplifications in the parameters, we have:

$$b_s = \frac{b_{s-1} + h_1 + h_2.vh_{-1} + h_3.b_{f_{-1}}}{h_4 + h_5.vh_{-1}} \qquad \text{(SF.1)}$$

Where: $h_1 = -\sigma_1.\psi_1.(\rho + \gamma_0 - \gamma_2 r) - \gamma_2.r$; $h_2 = -\sigma_1.\psi_1.\alpha_2 + \sigma_2.r$; $h_3 = \sigma_3.r$; $h_4 = 1 + \gamma_0 + \gamma_1.\psi_1.(\rho + \gamma_0 - \gamma_2 r) - \gamma_2 r$; $h_5 = \gamma_1.\psi_1.\alpha_2$

Equation SF1 states that the total securities issued (normalized) positively depend on the total securities issued in the previous period and corporate debt normalizes. The relationship with inherited wealth is positive or negative depending on the set of parameters used.

$$b_f = \frac{v_1.b_{f_{-1}} + v_2 + v_3.vh_{-1}}{v_4 + v_5.vh_{-1}} \qquad \text{(SF.2)}$$

Where: $v_1 = 1 + (1-d).r;\ v_2 = \gamma_0 - \gamma_2 r + \psi_2.\psi_1.(\rho + \gamma_0 - \gamma_2 r); v_3 = \psi_2.\psi_1.\alpha_2; v_4 = 1 + \gamma_0 + \gamma_1.\psi_1.(\rho + \gamma_0 - \gamma_2 r) - \gamma_2 r;\ v_5 = \gamma_1.\psi_1.\alpha_2$

Equation SF2 states that (normalized) corporate debt has a positive relationship with inherited (normalized) corporate debt and a relationship that may be negative or positive with normalized wealth.

$$vh = \frac{q_1.vh_{-1} + q_2 - q_3.b_{f_{-1}}}{(q_4 + q_5.vh_{-1})} \qquad \text{(SF.3)}$$

Where: $q_1 = k_1 + k_2.r + k_3.\psi_1.\alpha_2; q_2 = k_3.\psi_1.(\rho + \gamma_0 - \gamma_2 r);\ q_3 = k_4.r; q_4 = 1 + \gamma_0 + \gamma_1.\psi_1.(\rho + \gamma_0 - \gamma_2 r) - \gamma_2 r; q_5 = \gamma_1.\psi_1.\alpha_2$

Equation SF3 states that there is a negative relationship between normalized wealth and corporate debt. The relation with the wealth normalized and the lagged wealth normalized can be positive or negative depending on the set of parameters used.

7.2. The long-run equilibrium (Steady State):

Considering that in steady state the normalized stocks do not change, we have: $b_s = b_{s-1} = b_s^*;\ b_f = b_{f-1} = b_f^*; vh = vh_{-1} = vh^*$. Putting this condition into the three equations above, we then have the following equations:

$$b_s^* = \frac{h_1 + h_2.vh^* + h_3.b_f^*}{h_4 + h_5.vh^* - 1} \qquad \text{(ST.1)}$$

$$b_f^* = \frac{v_2 + v_3.vh^*}{v_4 + v_5.vh^* - v_1} \qquad \text{(ST.2)}$$

$$vh^* = \frac{q_1.vh^* + q_2 - q_3.b_f^*}{(q_4 + q_5.vh^*)} \qquad \text{(ST.3)}$$

Notice that we now have three equations that give us fixed points for the steady state. However, (ST1) depends on b_f^* and vh^*. The (ST2) depends only on vh^*. (ST3) depends only on vh^* and b_f^*. There are a few ways to solve this system. The first of these is numerically through the Newton-Raphson method. However, this is very sensitive to the initial guess. Another way to find the values is by plotting ST2 and ST3 for a given set of parameters and analyzing the points where the curves meet.

Finally, we have another alternative. We can use the above information. Knowing that b_f^* depends on vh^* and that vh^* depends on vh^* and b_f^* we can then solve for both variables by substitution and with the found value, use for to find b_s^*.

Let us then take the inverse function of vh^*:

$$f(vh^*)^{-1} = \frac{q_2 - (q_4 - q_1).vh^* - q_5.vh^{*2}}{q_3}$$

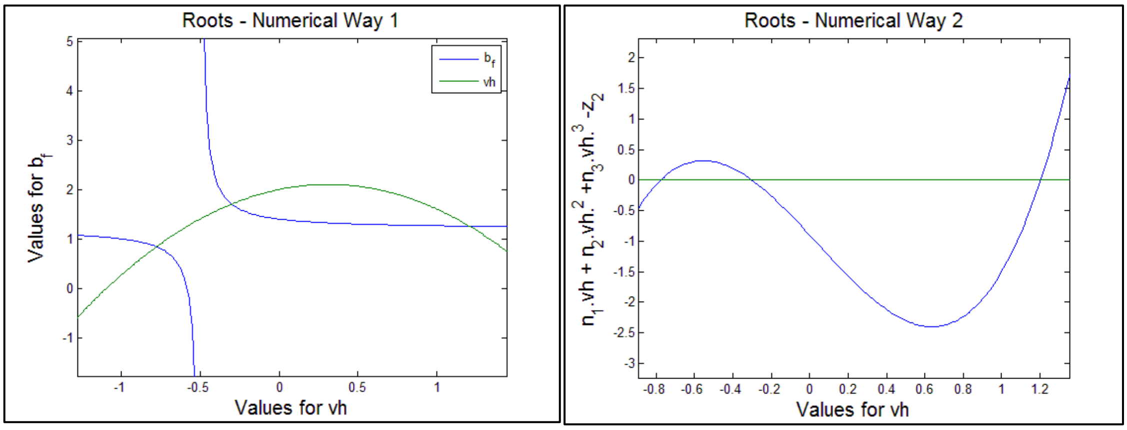
By inserting the inverse function in bf, we have:

$$n_1.vh^* + n_2.vh^{*2} + n_3.vh^{*3} - z_2 = 0$$

Where: $n_1 = z_1.(q_4 - q_1) - z_3$; $n_2 = z_1.q_5 + (q_4 - q_1).v_5$; $n_3 = v_5.q_5$

Figure 5 below presents the 2 previously mentioned methods (except Newton-Raphson). In the left part we have the plot of the curves b_f^* and vh^* for a set of parameters. In the right part, we have the plot of the polynomial from the substitution of equations. As can be seen, the values of vh^* that are roots of the polynomial are the values that cross the curves in the figure to the right.

Figure 5: The two ways to find the roots numerically for the steady-state



7.3. The Steady-State Stability:

In the last section we present the system composed of three equations that show the dynamics of the model. These three equations form a system of non-linear difference equations. Being a 3 x 3 nonlinear system, we will assume that in the neighborhood of the steady state it can be linearized and thus to reach stability it is sufficient to identify the eigenvalues of the Jacobian matrix[7].

The Steady-State Jacobian Matrix:

$$J(v^*) = \begin{bmatrix} \frac{1}{h_4 + h_5.vh^*} & \frac{h_3}{h_4 + h_5.vh^*} & \frac{h_2}{h_4 + h_5.vh^*} - \frac{h_5.\left(b_s^* + h_1 + b_f^*.h_3 + h_2.vh^*\right)}{(h_4 + h_5.vh^*)^2} \\ 0 & \frac{v_1}{v_4 + v_5.vh^*} & \frac{v_3}{v_4 + v_5.vh^*} - \frac{v_5.\left(v_2 + b_f^*.v_1 + v_3.vh^*\right)}{(v_4 + v_5.vh^*)^2} \\ 0 & -\frac{q_3}{q_4 + q_5.vh^*} & \frac{q_1}{q_4 + q_5.vh^*} - \frac{q_5.\left(q_2 - b_f^*.q_3 + q_1.vh^*\right)}{(q_4 + q_5.vh^*)^2} \end{bmatrix}$$

If the value of the real part of all eigenvalues is less than one, we have met the sufficiency condition for a stable fixed point.

Since it is difficult to define analytically the stability conditions, we performed a numerical algorithm to analyze the stability. This is what we call Method 1 (M1) and it was done as follows: parameters for the disaggregated model were randomly generated in an exhaustive way. Then these were aggregated and found steady state roots[8] for vh^* and by substitution found b_f^* and b_s^*.

After that, having the values of b_f^*, b_s^*, vh^* and the parameters, we introduce the values in the jacobian matrix and calculate the eigenvalue. If the values of the three calculated eigenvalues were less than one, the point was defined as stable. Otherwise, unstable. Figures 6, 7 and 8 below show the simulation results for 10^8 random data simulations.

[7] The steady-state stability of the non-linear multidimensional model is investigated by its linearization in the proximity of steady state. The steady-state fixed point will be stable if all eigenvalues of the Jacobian matrix have modules smaller than one. Stability is granted by the Hartman-Grobman theorem.

[8] Using equations ST1, ST2 and ST3 and a function to numerically finding polynomial roots.

Figure 6: Stability for Steady-State values of $\boldsymbol{b_s^*}$ and $\boldsymbol{b_f^*}$

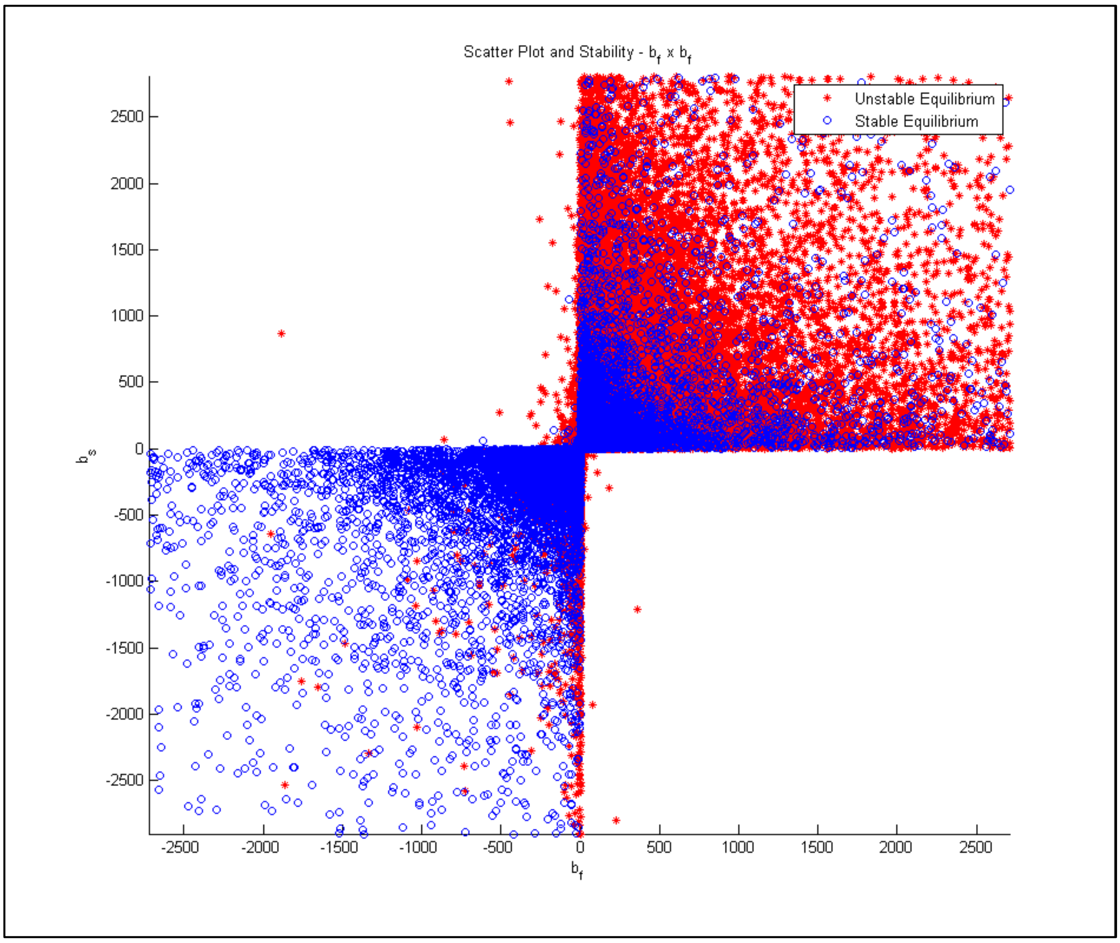


Figure 7 shows the pairs of values found for steady-state equilibrium for b_s^* and b_f^*. The most likely combination of pairs of values are: the negative values for b_s^* with negative values for b_s^* and positive values for b_s^* with positive values for b_f^*. Being that when both were positive, there is a large sample set that presents unstable behavior for very high values of b_s^* and b_f^*. When the values of b_s^* and b_f^* were small (less than 1000), the most likely behavior is stable equilibrium. Few values were found in other quadrants.

Figure 7: Stability for Steady-State values of $\boldsymbol{b_s^*}$ and $\boldsymbol{vh^*}$

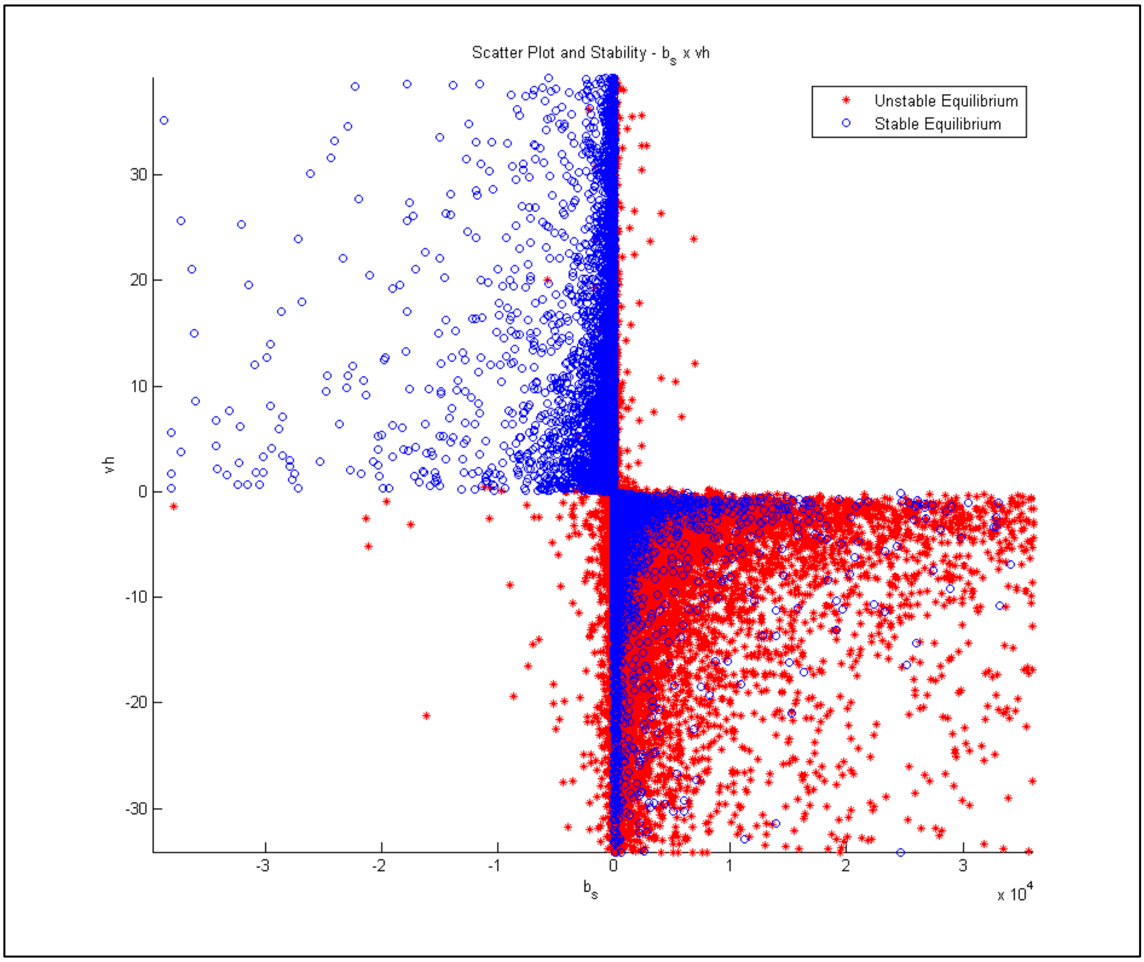


Figure 8 shows the results for b_s^* and vh^*. Most of the equilibrium values found are in the quadrants where vh^* is positive being b_s^* negative and in the quadrant where b_s^* is positive and vh^* is negative. No stable equilibrium value was found having the combination of negative b_s^* with negative vh^* or positive b_s^* with positive vh^*. In the quadrant where vh^* is positive and b_s^* negative, the almost absolute majority of the values found is of stable equilibrium. The pairs of small values for b_s^* (positive) and vh^* (negative) were mostly stable. When large, they presented behaviors mostly unstable.

Figure 8: Stability for Steady-State values of b_f^* and vh^*:

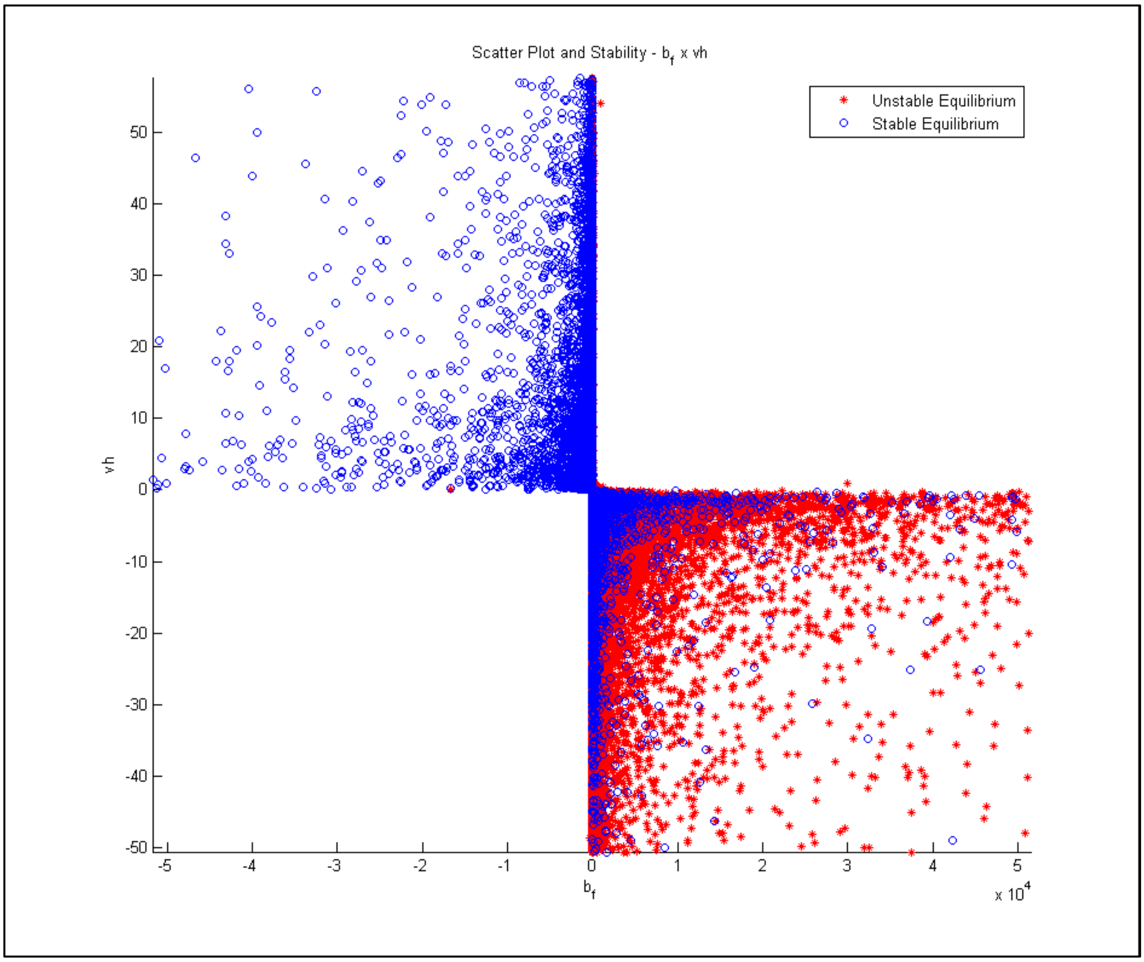
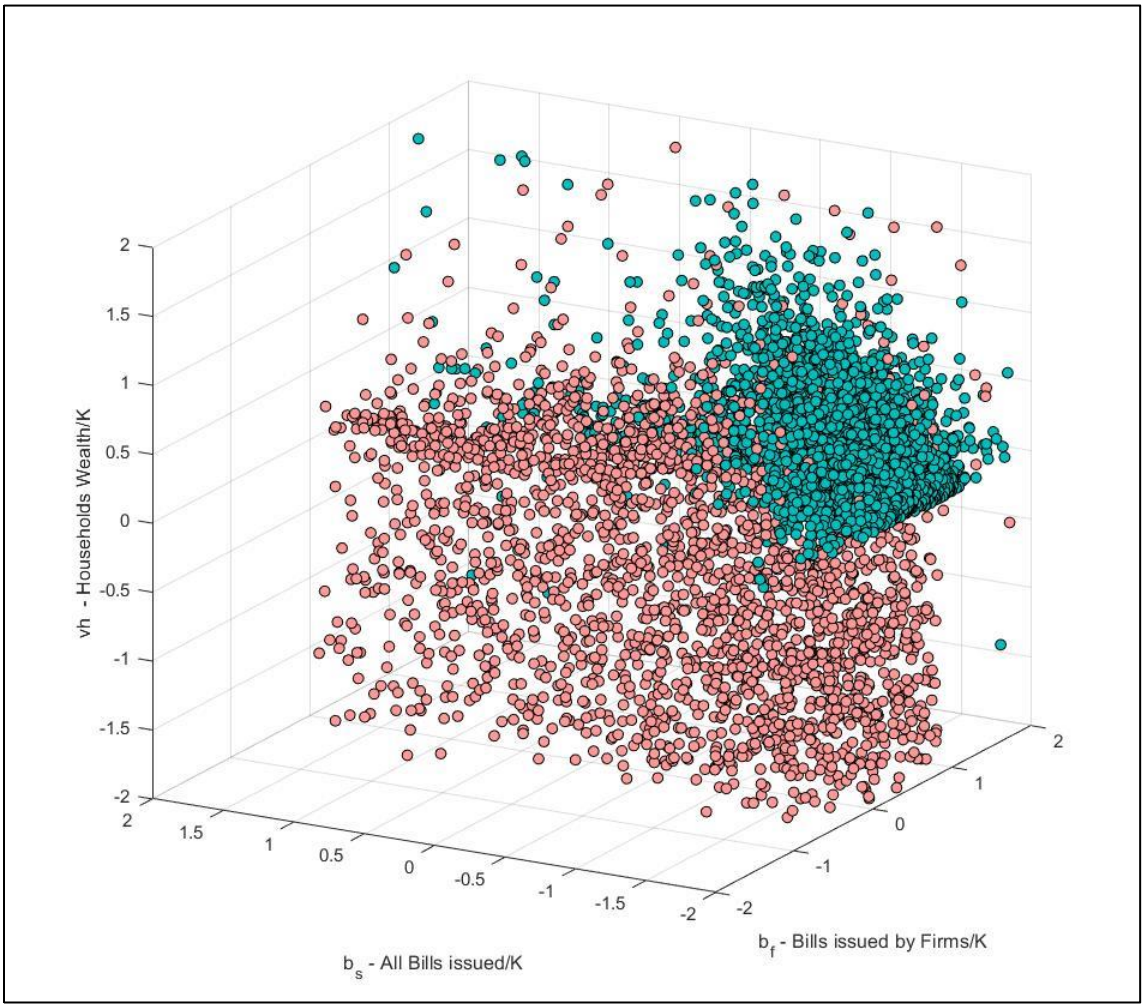


Figure 8 shows the pairs of values found for bf and vh^*. Most of the values found are in the quadrants where vh^* is positive with negative b_f^* and in the quadrant where vh^* is negative and b_f^* is positive. No values were found (or very low probability to found) in the vh^* negative quadrants with negative b_f^* or positive vh^* with positive b_f^*. Almost all values in the vh^* positive and b_f^* negative quadrant were stable. Small values in the quadrant b_f^* positive and vh^* negative were stable and large values were unstable.

Figure 9: 3D Stability for Steady-State values of b_f^*, b_s^* and vh^*:



After analyzing the three figures of scatter plot and stability, we can identify that the simple resolution of the model aimed at finding steady state values and stability are insufficient to find results that make economic sense. A clear result of this is the locally stable values for simultaneously negative b_f^* and b_s^*.

In this sense, we made use of what we call method 2 (M2), previously presented in section 2. The algorithm consists of generating parameters extensively and using them as an input for reduced version of the model given by SF1, SF2 and SF3. The system is solved by a modified Gauss-Seidel algorithm[9] and the values for Δvh, Δb_f and Δb_s are stored in the final simulation period. The data are filtered for which the value is less than

[9] Instead of breaking the algorithm with a tolerance value for convergence, the convergence time is set to a predetermined value. Further details on the Gauss-Seidel algorithm can be obtained in Grasselli and Pelinovsky (2008).

10^{-6} and it is assumed that it has already been convergence to the steady state. Having these final results, it is enough to take the opposite way and map the parameters responsible for such results.

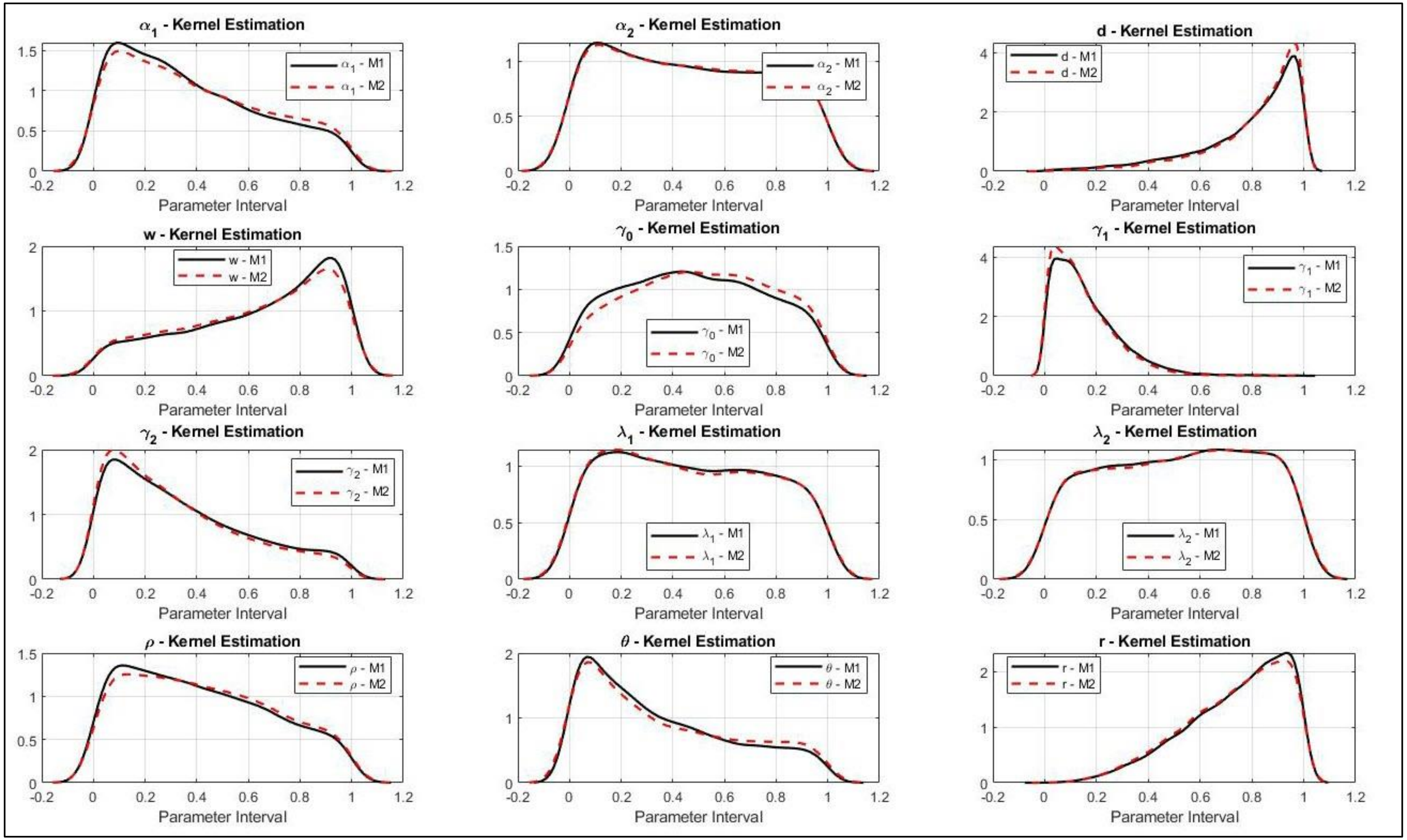
Finally, we can map the parameters that resulted in stable values for the steady state using M1 and M2. When these are stored, we perform a non-parametric probability density estimation using the Kernel method. These density curves can be visualized in figure 10 below.

Here it should be remembered that the model is non-linear and in the summarized form presented a polynomial of degree three which results in three possible steady state equilibria for each simulation. In some cases, there was only one stable equilibrium to which the model converged (other two unstable) and there were cases where there was more than one stable equilibrium. Thus, convergence gets stuck to the problem of the common initial value in equations to differences.

In addition, we can point out that all the results obtained by M2 are contained in M1, there being no single case in which a point in M2 arises that was not predicted in M1. The opposite is not true. There were stable equilibrium values in M2 that were not found in M1 and the reason can be explained by two situations: a) because the model has not yet converged to equilibrium; b) because of the initial value problem, the model can converge to another stable equilibrium.

Thus, we have that the algorithm M2 developed in this article has superior ability to map parameters that are of economic interest and this shows as an important escape route for the resolution of large SFC models.

Figure 10: Kernel Density Estimation for parameters obtained by M1 and M2



8. Final Remarks

The main objective of this article was to present the algorithm developed by Costa Santos (2017) in order to determine the steady-state values and stability properties of a simple Neo-Kaleckian SFC growth model. Initially we calibrate and set the initial conditions of the model in order to we run the baseline simulation, making possible to analyze the properties of the time path of endogenous variables and calculate their steady-state values. The general model dynamics and stability were analyzed after reducing the twenty equations of the previous model to a system of only 3 equations, where the endogenous variables (wealth, public debt and corporate debt) are normalized by capital stock.

The model is then simulated 10^8 times with random values for the parameters in order to determine the steady-state values of the endogenous variables and their stability properties. Finally, we calculated the values of the Jacobian matrix at steady state and in order to define the range of the parameters values that generate a stable and positive equilibrium values for the normalized wealth, public and corporate debt.

A final comparison among the methodologies pointed out that all results of M2 are in M1 while the opposite is not true. *The new methodology contributes to provide an escape route for the stability analysis of larger models*. In this case, it is not necessary to reduce the model. Simply use the M2 criteria to map the stable results and investigate their properties on a case-by-case basis.

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Appendix:

1. The Hartman-Grobman Theorem:

Being $F: R^N \to R^N$ A class C^1 diffeomorphism with a hyperbolic fixed point $\mathbf{v}^*$. Then there is a homeomorphism **h**, defined in some neighborhood U of the fixed point $\mathbf{v}^*$ such that, for all $\mathbf{v}^* \in$ U

$$\mathbf{h}\big(\mathbf{F}(\mathbf{v}^*)\big) = \mathbf{J}(\mathbf{v}^*).\,\mathbf{h}(\mathbf{v}^*)$$

That is, h takes orbits generated by the discrete nonlinear model

$$v_t = \boldsymbol{F}(v_{t-1})$$

In orbits of discrete linear model

$$\Delta\mathbf{v_t} = \mathbf{J}(\mathbf{v}^*).\,\Delta\mathbf{v_{t-1}}$$