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THE SOLUTION OF LINEAR DIFFERENCE MODELS UNDER RATIONAL EXPECTATIONS

By Olivier Jean Blanchard and Charles M. Kahn[1]

In his survey on rational expectations, R. Shiller indicates that the difficulty of obtaining explicit solutions for linear difference models under rational expectations may have hindered their use [**14**, p. 27]. The present paper attempts to remedy that problem by giving the explicit solution for an important subclass of the model Shiller refers to as the general linear difference model.

Section 1 presents the form of the model for which the solution is derived and shows how particular models can be put in this form.

Section 2 gives the solution together with the conditions for existence and uniqueness.

1. THE MODEL

The model is given by (1a), (1b), and (1c) as follows:

(1a) $$\begin{bmatrix} X_{t+1} \\ {}_tP_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t, \qquad X_{t=0} = X_0,$$

where X is an $(n \times 1)$ vector of variables predetermined at t; P is an $(m \times 1)$ vector of variables non-predetermined at t; Z is a $(k \times 1)$ vector of exogenous variables; ${}_tP_{t+1}$ is the agents' expectation of P_{t+1}, held at t; A, γ are $(n+m) \times (n+m)$ and $(n+m) \times k$ matrices, respectively.

(1b) $$\quad {}_tP_{t+1} = E(P_{t+1}|\Omega_t)$$

where $E(\cdot)$ is the mathematical expectation operator; Ω_t is the information set at t; $\Omega_t \supseteq \Omega_{t-1}$; Ω_t includes at least past and current values of X, P, Z (it may include other exogenous variables than Z; it may include future values of exogenous variables).

(1c) $$\quad \forall t \; \exists \bar{Z}_t \in \mathbb{R}^k, \qquad \theta_t \in \mathbb{R} \qquad \text{such that}$$

$$-(1+i)^{\theta_t}\bar{Z}_t \leqslant E(Z_{t+i}|\Omega_t) \leqslant (1+i)^{\theta_t}\bar{Z}_t \qquad \forall i \geqslant 0.$$

Equation (1a) describes the structural model. The difference between predetermined and non-predetermined variables is extremely important. A predetermined variable is a function only of variables known at time t, that is of variables in Ω_t, so that $X_{t+1} = {}_tX_{t+1}$ whatever the realization of the variables in Ω_{t+1}. A non-predetermined variable P_{t+1} can be a function of any variable in Ω_{t+1}, so that we can conclude that $P_{t+1} = {}_tP_{t+1}$ only if the realizations of all variables in Ω_{t+1} are equal to their expectations conditional on Ω_t.

The structural model imposes the restriction that all agents at a given time have the same information, so that "agents' expectation" has a precise meaning. As Example B below shows, the "first order" form is not restrictive: models of higher order can be reduced to this form.

Equation (1b) defines rational expectations. Equation (1b) excludes the possibility that agents know the values of endogenous variables but not the values of the exogenous variables: in such a case, endogenous variables convey information on exogenous variables; such cases require a different treatment (see Futia [**8**]). Condition (1c) simply requires that the exogenous variables Z do not "explode too fast." In effect it rules out exponential growth of the expectation of Z_{t+i}, held at time t.

The following examples show how particular models can be put in the required form.

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EXAMPLE A (A Model of Growth with Money): Consider, for example, the structure of the model presented by Sidrauski [15]; savings and thus capital accumulation depend on disposable income, which itself depends on the capital stock K_t, real money balances, $M_t - P_t$ (where M and P denote logarithms), and the expected rate of inflation ($_tP_{t+1} - P_t$):

$$K_{t+1} - K_t = f(K_t, M_t - P_t, {}_tP_{t+1} - P_t).$$

From asset market equilibrium, there is another relation between the real money stock, the capital stock, and the expected rate of inflation:

$$M_t - P_t = g(K_t, {}_tP_{t+1} - P_t).$$

The model can be written, linearized around its steady state, as

$$\begin{bmatrix} K_{t+1} \\ {}_tP_{t+1} \end{bmatrix} = A \begin{bmatrix} K_t \\ P_t \end{bmatrix} + \gamma M_t.$$

K_t, the capital stock, is predetermined at time t. P_t, the price level, is not. (The model solved by Sidrauski assumes adaptive, not rational, expectations.)

EXAMPLE B: Models with lagged variables or current expectations of variables more than one period ahead present no particular problem. Consider the following equation to which no economic interpretation will be given:

$$Y_t + \alpha Y_{t-2} + \beta_t Y_{t+2} = Z_t.$$

Define

$$XI_t \equiv Y_{t-1}; \qquad X2_t \equiv XI_{t-1} = Y_{t-2};$$

$$P_t \equiv {}_tY_{t+1} \Rightarrow {}_tP_{t+1} = {}_t({}_{t+1}Y_{t+2}) = {}_tY_{t+2}.$$

The above equation may be rewritten as

$$\begin{bmatrix} XI_{t+1} \\ X2_{t+1} \\ {}_tY_{t+1} \\ {}_tP_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\alpha/\beta & -1/\beta & 0 \end{bmatrix} \begin{bmatrix} XI_t \\ X2_t \\ Y_t \\ P_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\beta \end{bmatrix} Z_t.$$

XI_t, $X2_t$ are predetermined at t; Y_t and P_t are not.

An example of the reduction of a medium size empirical macroeconometric model to a model of form (1) can be found in Blanchard [2].

EXAMPLE C: Models which include past expectations of current and future variables on the other hand may be such that they cannot be reduced to form (1). The simplest example is

$$Y_t - a_{t-1}Y_t - Z_t.$$

This is in effect a "zeroth order" difference equation which cannot be put in the "first order" form (1). The same difficulty may arise in more complex models:

$$P_t = \alpha ({}_{t-1}P_{t+1} - {}_{t-1}P_t) + \varepsilon_t.$$

(This may be interpreted as the equilibrium condition of an asset market model.)

EXAMPLE D: This last example shows however that some models with lagged expectations of present and future variables can be put in form (1). This model can be interpreted

as a multiplier accelerator model:

$$Y_t = C_t + I_t + G_t,$$

$$C_t = \alpha(Y_t + {}_tY_{t+1}) + \varepsilon_t, \qquad \alpha > 0,$$

$$I_t = \beta({}_tY_{t+1} - {}_{t-1}Y_t) + \eta_t, \qquad \beta > 0,$$

where all symbols are standard and ε_t, η_t are disturbances. Solving the equilibrium condition and defining $X_t = {}_{t-1}Y_t \Rightarrow X_{t+1} = {}_tY_{t+1}$ gives

$$\begin{bmatrix} X_{t+1} \\ {}_tY_{t+1} \end{bmatrix} = \frac{1}{\alpha + \beta} \begin{bmatrix} \beta & (1-\alpha) \\ \beta & (1-\alpha) \end{bmatrix} \begin{bmatrix} X_t \\ Y_t \end{bmatrix} + \frac{1}{\alpha + \beta} \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} G_t \\ \varepsilon_t \\ \eta_t \end{bmatrix}.$$

The matrix A is in this case singular. This example also shows the absence of necessary connection between "real," "nominal" and "predetermined," "non-predetermined."

2. THE SOLUTION

A solution (X_t, P_t) is a sequence of functions of variables in Ω_t which satisfies (1) for all possible realizations of these variables.

In a manner analogous to (1c), we also require that expectations of X_t and P_t do not explode. More precisely,

$$\forall t \; \exists \begin{bmatrix} \bar{X}_t \\ \bar{P}_t \end{bmatrix} \in \mathbb{R}^{n+m}, \quad \sigma_t \in \mathbb{R} \quad \text{such that}$$

$$-(1+i)^{\sigma_t} \begin{bmatrix} \bar{X}_t \\ \bar{P}_t \end{bmatrix} \le \begin{bmatrix} E(X_{t+i}|\Omega_t) \\ E(P_{t+i}|\Omega_t) \end{bmatrix} \le (1+i)^{\sigma_t} \begin{bmatrix} \bar{X}_t \\ \bar{P}_t \end{bmatrix}, \qquad \forall i \ge 0.$$

This condition in effect rules out exponential growth of the expectation of X_{t+i} and P_{t+i} held at time t. (This in particular rules out "bubbles" of the sort considered by Flood and Garber [7].)

Our strategy is to simplify the model by transforming it into canonical form, following Vaughan [18]. Thus A is first transformed into Jordan canonical form (see, for example, Halmos [10, pp. 112–115]):

$$A = C^{-1}JC$$

where the diagonal elements of J, which are the eigenvalues of A, are ordered by increasing absolute value.

J is further decomposed as

$$J = \begin{bmatrix} J_1 & & 0 \\ (\bar{n} \times \bar{n}) & & \\ 0 & & J_2 \\ & & (\bar{m} \times \bar{m}) \end{bmatrix}$$

where J is partitioned so that all eigenvalues of J_1 are on or inside the unit circle, all eigenvalues of J_2 are outside the unit circle. Note that so far nothing has been said about the relation between m and $\bar{m}$.

C, C^{-1}, and γ are decomposed accordingly:

$$C \equiv \begin{bmatrix} C_{11} & C_{12} \\ (\bar{n} \times n) & (\bar{n} \times m) \\ C_{21} & C_{22} \\ (\bar{m} \times n) & (\bar{m} \times m) \end{bmatrix}; \quad C^{-1} \equiv \begin{bmatrix} B_{11} & B_{12} \\ (n \times \bar{n}) & (n \times \bar{m}) \\ B_{21} & B_{22} \\ (m \times \bar{n}) & (m \times \bar{m}) \end{bmatrix}; \quad \gamma = \begin{bmatrix} \gamma_1 \\ (\bar{n} \times k) \\ \gamma_2 \\ (\bar{m} \times k) \end{bmatrix}.$$

We may now summarize the results by three propositions. In these propositions, the rank of C_{22} will be assumed to be full, i.e. $\rho(C_{22}) = \min(\bar{m}, m)$. This implies that B_{11} is also of full rank. The propositions can be extended when this assumption is relaxed.

PROPOSITION 1: *If $\bar{m} = m$, i.e., if the number of eigenvalues of A outside the unit circle is equal to the number of non-predetermined variables, then there exists a unique solution.*

This solution is "forward looking" (Sargent and Wallace [12], Shiller [14], Blanchard [1]) in the following sense: the non-predetermined variables depend on the past only through its effect on the current predetermined variables. This solution is

$$(2) \qquad X_t = X_0, \qquad \text{for } t = 0,$$

$$= B_{11}J_1B_{11}^{-1}X_{t-1} + \gamma_1 Z_{t-1}$$

$$- (B_{11}J_1C_{12} + B_{12}J_2C_{22})C_{22}^{-1} \sum_{i=0}^{\infty} J_2^{-i-1}(C_{21}\gamma_1 + C_{22}\gamma_2)E(Z_{t+i-1}|\Omega_{t-1}),$$

$$\text{for } t > 0,$$

$$(3) \qquad P_t = -C_{22}^{-1}C_{21}X_t - C_{22}^{-1} \sum_{i=0}^{\infty} J_2^{-i-1}(C_{21}\gamma_1 + C_{22}\gamma_2)E(Z_{t+i}|\Omega_t), \qquad \text{for } t \geq 0.$$

It can be solved recursively to give the final form solution:

$$(4) \qquad X_t = -\sum_{j=1}^{t} B_{11}J_1^{j-1}(B_{11}^{-1}B_{12} - J_1B_{11}^{-1}B_{12}J_2^{-1}) \sum_{i=0}^{\infty} J_2^{-i}(C_{21}\gamma_1 + C_{22}\gamma_2)E(Z_{t-j+i}|\Omega_{t-j})$$

$$+ \sum_{j=1}^{t} B_{11}J_1^{j-1}B_{11}^{-1}\gamma_1 Z_{t-j} + B_{11}J_1^{t}B_{11}^{-1}X_0, \qquad \text{for } t > 0,$$

$$(5) \qquad P_t = -\sum_{j=1}^{t} B_{21}J_1^{j-1}(B_{11}^{-1}B_{12} - J_1B_{11}^{-1}B_{12}J_2^{-1}) \sum_{i=0}^{\infty} J_2^{-i}(C_{21}\gamma_1 + C_{22}\gamma_2)E(Z_{t-j+i}|\Omega_{t-j})$$

$$- \sum_{i=0}^{\infty} C_{22}^{-1}J_2^{-i-1}(C_{21}\gamma_1 + C_{22}\gamma_2)E(Z_{t+i}|\Omega_t)$$

$$+ \sum_{j=1}^{t} B_{21}J_1^{j-1}B_{11}^{-1}\gamma_1 Z_{t-j} + B_{21}J_1^{t}B_{11}^{-1}X_0, \qquad \text{for } t \geq 0.$$

PROPOSITION 2: *If $\bar{m} > m$, i.e. if the number of eigenvalues outside the unit circle exceeds the number of non-predetermined variables, there is no solution satisfying both (1) and the non-explosion condition.*

PROPOSITION 3: *If $\bar{m} < m$, i.e. if the number of eigenvalues outside the unit circle is less than the number of non-predetermined variables, there is an infinity of solutions.*
A solution may in particular be such that the non-predetermined variables depend on the past directly and not only through its effect on the currently predetermined variables.
A solution may also be such that a variable not belonging to Z directly affects X and P, i.e. that such a variable may be directly included in the functional form of the solution. Note that this possibility is excluded when $m = \bar{m}$. (Of course, if a variable provides information as to values of future Z's, it will affect indirectly X and P through expectations of future Z's.)

More precise statements of the last two propositions, together with a sketch of proofs are given in the Appendix.

We conclude with a series of remarks.

How likely are we to have $\bar{m} = m$ in a particular model?

It may be that the system described by (1) is just the set of necessary conditions for maximization of a quadratic function subject to linear constraints. In this case the matrix A will have more structure than we have imposed here, and the condition $\bar{m} = m$ will always hold. (See Hansen and Sargent [11].)

The condition $\bar{m} = m$ is also clearly related to the strict saddle point property discussed in the context of growth models: the above proposition states in effect that a unique solution will exist if and only if A has the strict saddle point property. Hence we know that many models have this property. Most recent macroeconomic models also satisfy $\bar{m} = m$; this is the case for Sargent and Wallace [13] or the larger size models by Hall [9], Taylor [17], and Blanchard [2], for example.

Finally, we know that examples can be constructed which do not satisfy $\bar{m} = m$. Models in which $\bar{m} < m$ have been constructed by—in increasing order of complexity—Blanchard [1], Shiller [14], Taylor [16], Calvo [5], and Burmeister, Caton, Dobell, and Ross [3], among others. Taylor in particular has pointed out the non-uniqueness and the possible presence of irrelevant variables in the solution.

If a model is to be used for simulations, the existence of a unique solution is easily checked by computing the eigenvalues of A. Simulations should use the recursive formulas (2) and (3) rather than the final form solution (4) and (5). A computer algorithm corresponding to (2) and (3) is available upon request.

Because they require computing the roots of an $n + m$th order polynomial, (2) and (3) are in general analytically intractable. The case where n and m are equal to one, i.e. the case where there is one predetermined and one non-predetermined variable appears often (Fischer [6], for example) and is easy to handle analytically. Its solution is given here for convenience. Let

$$
\begin{matrix} A \\ (2 \times 2) \end{matrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \begin{matrix} \gamma \\ (2 \times k) \end{matrix} = \begin{bmatrix} \gamma_1 \\ (1 \times k) \\ \gamma_2 \\ (1 \times k) \end{bmatrix},
$$

let λ_1, λ_2 be the eigenvalues of A, $|\lambda_1| < 1$, $|\lambda_2| > 1$. Define $\mu \equiv (\lambda_1 - a_{11})\lambda_1 - a_{12}\lambda_2$. Then, a unique solution exists and is given by

$$x_t = x_0, \qquad \text{for } t = 0,$$

$$= \lambda_1 x_{t-1} + \gamma_1 Z_{t-1} + \mu \sum_{i=0}^{\infty} \lambda_2^{-i-1} E(Z_{t+i-1} | \Omega_{t-1}), \qquad \text{for } t > 0,$$

$$p_t = a_{12}^{-1} \left[(\lambda_1 - a_{11}) x_t + \mu \sum_{i=0}^{\infty} \lambda_2^{-i-1} E(Z_{t+i} | \Omega_t) \right], \qquad \text{for } t \geq 0.$$

CONCLUSION

We have derived explicit solutions, and conditions for the existence and uniqueness of those solutions, for models of form (1). Although the class of models reducible to this form includes most existing models, Example C demonstrates that there exist models which cannot be reduced to this form. Thus there remain two open questions: the characterization of the class of models not reducible to (1) and the extension of this method to cover models in that class.

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APPENDIX

Consider the system given by (1), at time $t + i$. Take expectations on both sides, conditional on Ω_t. As $\Omega_t \subseteq \Omega_{t+i}$, this gives

(A1) $\qquad \begin{bmatrix} {}_tX_{t+i+1} \\ {}_tP_{t+i+1} \end{bmatrix} = A \begin{bmatrix} {}_tX_{t+i} \\ {}_tP_{t+i} \end{bmatrix} + \gamma_t Z_{t+i}, \qquad \forall i \geq 0.$

Consider the transformation

$$\begin{bmatrix} Y_t \\ Q_t \end{bmatrix} = C \begin{bmatrix} X_t \\ P_t \end{bmatrix}, \qquad \text{where } C \text{ is defined in the text.}$$

Premultiplying both sides of (A1) by C, and using $A = C^{-1}JC$,

(A2) $\qquad \begin{bmatrix} {}_tY_{t+i+1} \\ {}_tQ_{t+i+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} {}_tY_{t+i} \\ {}_tQ_{t+i} \end{bmatrix} + C\gamma_t Z_{t+i} \qquad \forall i \geq 0.$

As C is invertible, knowledge of X_t and P_t is equivalent to knowledge of Y_t and Q_t: the transformation does not affect Ω_t. Also existence (uniqueness) of a solution in (A2) is equivalent to existence (uniqueness) of a solution in (A1).

Equation (A2) is composed of two subsystems. The first $\bar{n}$ lines give

(A3) $\qquad {}_tY_{t+i+1} = J_1 {}_tY_{t+i} + (C_{11}\gamma_1 + C_{12}\gamma_2)_t Z_{t+i}, \qquad \forall i \geq 0.$

By construction of J_1, this system is stable or borderline stable. The second subsystem however will, by construction of J_2, explode and violate the non-explosion condition unless:

(A4) $\qquad Q_t = -\sum_{i=0}^{\infty} J_2^{-i-1} (C_{21}\gamma_1 + C_{22}\gamma_2)_t Z_{t+i}.$

(A4) uniquely determines Q_t. Thus existence and uniqueness of solutions depends on existence and uniqueness of the sequence of Y_t, Y_t has to satisfy (A3). Because (A2) is derived from (A1), a solution must however also satisfy two other conditions:

Consider the inverse transformation

$$\begin{bmatrix} X_t \\ P_t \end{bmatrix} = C^{-1} \begin{bmatrix} Y_t \\ Q_t \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} Y_t \\ Q_t \end{bmatrix},$$

where C^{-1} was defined in the text. Expanding the first n lines at time $t = 0$,

(A5) $\qquad X_0 = B_{11}Y_0 + B_{12}Q_0.$

Thus initial conditions X_0 impose restrictions on Y_0. The first n lines also imply:

$$X_{t+1} - {}_tX_{t+1} = B_{11}(Y_{t+1} - {}_tY_{t+1}) + B_{12}(Q_{t+1} - {}_tQ_{t+1}).$$

As X_t is predetermined $X_{t+1} = {}_tX_{t+1}$. This imposes the following relation on Y and Q:

(A6) $\qquad 0 = B_{11}(Y_{t+1} - {}_tY_{t+1}) + B_{12}(Q_{t+1} - {}_tQ_{t+1}).$

In what follows, we assume that B_{11} is of full rank, i.e. that $\rho(B_{11}) = \min(m, \bar{m})$; this is equivalent to assuming that C_{22} is of full rank. The extension to the case where B_{11} is not of full rank is straightforward and tedious. If $\bar{m} = m$, then $\exists B_{11}^{-1}$. From (A4), Q_0 is determined. From (A5), Y_0 is uniquely determined. From (A3), $_0Y_1$ is determined; and Y_1 is determined from (A6). The system is solved recursively. The sequence of X and P is in turn obtained by using the C^{-1} transformation. Tedious computation gives (2) and (5) in the text. This proves Proposition 1.

If $\bar{m} > m$, B_{11} imposes more than $\bar{n}$ restrictions on Y_0 in (A5). (A5) is overdetermined and thus has almost always no solution. If Y_0 does not exist, then P_0 does not exist. This proves Proposition 2.

If $\bar{m} < m$, (A5) is underdetermined. In addition (A6) no longer uniquely determines Y_{t+1} given $_tY_{t+1}$. In general $Y_{t+1} = {}_tY_{t+1} + W_{t+1}$ satisfies (A6), where W_{t+1} is any random variable such that

(A7) $\qquad W_{t+1} \in \Omega_{t+1}; \qquad {}_{t-j}W_{t+1} = 0 \quad \forall j \geq 0; \qquad B_{11}W_t = B_{12}({}_{t-1}Q_t - Q_t).$

Because B_{11} is not invertible, W_t may include variables other than Z. Thus the general solution is (A4)

for Q_t, and

$$Y_t = J_1 Y_{t-1} + (C_{11}\gamma_1 + C_{12}\gamma_2)Z_{t-1} + W_t$$

where W_t satisfies (A7) and Y_0 satisfies (A5). Again the C^{-1} transformation can be used to solve for X_t and P_t. This proves Proposition 3.

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