Sunspots and Cycles
Author(s): Costas Azariadis and Roger Guesnerie
Source: *The Review of Economic Studies*, Vol. 53, No. 5 (Oct., 1986), pp. 725-737
Published by: Oxford University Press
Stable URL: https://www.jstor.org/stable/2297716
Accessed: 30-05-2019 23:04 UTC

Review of Economic Studies (1986), LIII, 725-737 0034-6527/86/00450725$02.00

Sunspots and Cycles

COSTAS AZARIADIS
University of Pennsylvania

and

ROGER GUESNERIE
EHESS and ENPC, Paris

Because sunspot equilibria seem to be of central importance for an understanding of rational expectations, we seek here to characterize completely a limited class of sunspot equilibria (stationary ones with two possible natural events) in the simplest overlapping generations model of production. We present a sufficient condition for the existence of stationary sunspot equilibria, examine how these are related to strictly periodic equilibria of the same order, and investigate how deterministic stationary equilibria bifurcate to stationary sunspot equilibria. A concluding section examines how our results survive in more general settings.

I. INTRODUCTION

Models of rational expectations (Lucas (1972), Radner (1979), Grossman and Stiglitz (1976)) typically determine prices on the basis of the intrinsic data of an economy—preferences, endowments, production sets. Price uncertainty in this framework is merely a reflection of randomness in the underlying intrinsic structure itself, and is not affected by "extrinsic" uncertainty, i.e. by events unrelated to economic "fundamentals".

From recent work undertaken in the framework of the overlapping generations model (Shell (1977), Azariadis (1981), Cass-Shell (1983)) we know now that we can construct examples of rational expectations equilibria with random prices and no intrinsic uncertainty. The randomness of these prices is due entirely to the beliefs that individuals hold about their environment. If these beliefs are shared by everyone, price randomness may be self-fulfilling and will not of necessity dissipate, even asymptotically, as individuals accumulate more observations. We call this phenomenon a "sunspot" equilibrium.

The meaning of sunspot equilibria is still open to interpretation. One may view "sunspots" as a convenient label for a host of psychological factors (animal spirits, fears, Bayesian learning theories, etc.) that are unrelated to the preferences, endowment or production set of any individual, and yet come to influence the forecasts and actions of economic decision-makers. And the reading of economic historians may suggest that these factors have some pertinence for the explanation of phenomena like the Dutch tulip mania in the seventeenth century and the Great Depression in our own. Whatever it may denote, the concept of sunspot equilibrium seems to be of central importance for a thorough understanding of rational expectations as an equilibrium construct. The general purpose of this paper, which follows a previous attempt by Azariadis and Guesnerie (1982), is to contribute towards the clarification of that construct, and especially of rational expectations equilibria in non-linear dynamic economies.

More precisely, we seek completely to characterize a limited class of sunspot equilibria (stationary ones of order two, i.e. with two possible events or states of nature) in a simple overlapping generation (OLG) model of identical households that consume a single produced good. This rather modest objective deserves some comment.

The study of rational expectations equilibria in *linear* systems has borne fruit only recently, through the work of Taylor (1977), Shiller (1978), and Gouriéroux, Laffont and Montfort (1982). The economics of the *non-linear* case is bound to be more complex; the present state of the mathematical theory of non-linear systems suggests, moreover, that it may be wise to begin with problems of low dimension.

Sunspot phenomena, of course, are not necessarily dynamical; the related concept of "correlated equilibrium" (see Maskin and Tirole (1985)), for instance, does not require the passage of time. But these ideas make more sense when stable beliefs are supported as long-run equilibria of an open-ended economy like ours. Furthermore, the simple OLG model has been prominent in the recent revival of dynamical macroeconomics, of which one example is the work of Grandmont (1985). We choose, in other words, to study an economy whose perfect foresight equilibria are well understood, and that understanding includes periodic equilibria.

Stationarity in its extended sense is important for two reasons: because stable beliefs are likely to be the asymptotic outcome of many well-defined learning processes; and because understanding stationary sunspot equilibria is a prerequisite towards understanding dynamical sunspot phenomena.

Our first result (Theorem 1) expresses formally a rather straightforward connection between sunspots and cycles. We continue with a sufficient condition (Theorem 2) for the existence of sunspot equilibria. This condition, which bears upon the stochastic characteristics of extrinsic uncertainty and the shape of the saving function, describes a class of economies in which sunspot equilibria exist. As it turns out, the same condition implies local asymptotic stability of the golden rule and existence of periodic equilibria of order 2. The reasons for this connection are elucidated in Theorem 3 which describes how stationary equilibria bifurcate to sunspot equilibria.

These results led us to investigate in greater depth the relationship between sunspot equilibria and cycles. The connection exhibited is surprisingly strong: Theorem 4 asserts that cycles of order 2 exist, if and only if, sunspot equilibria with two states exist.

Which of our results will survive in a more general model? The issue is taken up in the concluding section. All we need say here is that extensions from two to more than two natural events or from one to more than one type of household require relatively straightforward modifications of the methodology presented here; extensions to more than one physical commodity are the subject of a separate paper by Guesnerie (1985); and examples of non-stationary sunspot equilibria are provided in the appendix to this paper as well as in a related work by Peck (1984).

II. PERFECT FORESIGHT

The framework we shall be using is the overlapping generations model of fiat money with production, a simple reinterpretation of the related pure-exchange model examined by Samuelson (1958), Gale (1973) and others. Our results would not change in any essential way if we focused instead on pure exchange; we retain production here in order to preserve uniformity with earlier research on sunspots and cycles.

Time extends from one to infinity; at discrete points in time $t = 1, 2, \ldots,$ a fixed-size generation of identical individuals appears, lives for two periods, "youth" and "old age", and dies at $t+2$. Consumption occurs only in old age, production takes place only in youth. Each member of generation t is endowed with $e_1 > 0$ units of divisible leisure in youth, $e_2 > 0$ units of a single, perishable consumption good in old age. The only exception from this pattern is the very first generation that is born "old" at $t = 1$: each member of it is endowed with e_2 units of the consumption good and one unit of fiat "money", i.e. of an intrinsically worthless paper asset that will be the sole store of value in our economy.

Each member of the young generation may use a constant-returns-to scale technology to transform $n \in [0, e_1]$ units of his own leisure into $y \leqq n$ units of the perishable consumption good in order to purchase the store of value and finance in old age consumption in excess of e_2. The entire stock of the paper asset is then held by the old, provided it has positive value. All individuals are price takers and possess perfect foresight about future prices.

Individuals have preferences over intertemporal bundles. The utility of an individual born at t depends, first, upon the leisure he gives up at t or, equivalently, upon the amount of good he offers, y_t; second, upon his consumption c_{t+1} at $t+1$. The utility function denoted $u(c_{t+1}, y_t)$ is assumed to be monotone, twice continuously differentiable and strictly concave. We assume throughout this paper that consumption and leisure are (strict) *normal goods* and that young individuals would choose positive savings if confronted with a zero real rate of interest.

Community excess demand for the consumption good in period t is the sum $x_t - y_t$ of excess demands by the old (x_t) and by the young $(-y_t)$. In this simple model, x_t necessarily equals the purchasing power, $1/p_t$, of existing fiat money balances, so that one may define aggregate excess demand to be

$$D(p_t, p_{t+1}) = 1/p_t - s(p_t/p_{t+1}) \tag{1}$$

where

$$s(R) = \arg\max_{0 \leqq y \leqq e_1} u(e_2 + Ry, y) \tag{2}$$

is the savings function of the representative household.

A *competitive equilibrium with perfect foresight* is associated with a sequence $(p_t)_{t=1}^{\infty}$ of non-negative prices that satisfies $D(p_t, p_{t+1}) = 0$ for all t; or, equivalently, with a sequence $(m_t)_{t=1}^{\infty}$ of real money balances satisfying $D(1/m_t, 1/m_{t+1}) = 0$, where $m_t = 1/p_t$ by definition. Finding equilibria with perfect foresight is equivalent to "solving" the difference equation $D(p_t, p_{t+1}) = 0$ either backward or forward. A backward solution has the form $m_t = f(m_{t+1})$ and a forward one is of the form $m_{t+1} = \phi(m_t)$, where f and ϕ are known maps.

Of particular interest to this paper is the notion of periodicity. We call the sequence $(p_t)_{t=1}^{\infty}$ *a periodic competitive equilibrium of order* k (or k-cycle) if $p_t = p_{t+k}$ for $t = 1, 2, \ldots$ and $k \geqq 2$ while $p_t \neq p_{t+j}$ for all integers j in the interval $(0, k)$.

We state below without proof two useful results on competitive equilibria in general and on periodic equilibria in particular. Proposition 1 is well known and may be found, for example, in Cass Okuno and Zilcha (1979); Proposition 2 is due to Grandmont (1985).

Proposition 1. *If the individual's indifference map satisfies standard boundary assumptions and if consumption and leisure are normal goods, then a backward-looking competitive equilibrium exists and it is unique. In addition, if the current price is not "too small", a forward-looking competitive equilibrium exists as well but is not necessarily unique.*

Proposition 2. *If the monetary stationary equilibrium is locally dynamically stable, then a 2-cycle exists.*

Figure 1 illustrates: a competitive equilibrium is the sequence $(p_1, p_2, p_3, \ldots)$ in panel (a); the alternating price sequence $(1/\hat{m}, 1/f(\hat{m}), 1/\hat{m}, \ldots)$ in panel (b) is a two-cycle.

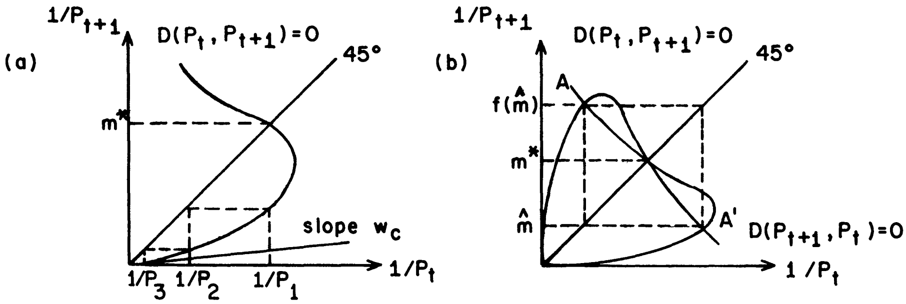

FIGURE 1

III. STATIONARY SUNSPOT EQUILIBRIA

Sunspot equilibria are rational expectations equilibria that are perfectly correlated with extraneous events or with factors other than the preferences, endowment and production set of any individual. Equilibria of this type are not necessarily stationary. We provide in the appendix an example of dynamical sunspot equilibria in which the impact of sunspots dissipates over time. Here we are only interested in stationary sunspots, for we wish to understand how the set of stationary equilibria is enlarged by the sunspot hypothesis. A natural complement of the present study would be to investigate how the set of *non-stationary* perfect-foresight equilibria is enlarged by the inclusion of sunspots, but that is outside the scope of this essay.

The event we are considering now is characterized by two values: either sunspot activity (a) or absence of sunspot activity (b). The occurrence of a and b is governed by a Markov process with the following stationary transition probability matrix

$$\Pi = \begin{pmatrix} \pi_{aa} & \pi_{ba} \\ \pi_{ab} & \pi_{bb} \end{pmatrix}. \tag{3}$$

For $i = a, b$ and $j = a, b$, an element π_{ij} of this matrix denotes the probability that sunspot activity will be i tomorrow given that it is j today.

Suppose now that all agents in the economy believe in a perfect and stationary correlation of future prices with sunspot activity: in other words, all individuals forecast future price to be $p = \phi(i)$, for $i = (a, b)$, if i occurs tomorrow. Loosely speaking, a stationary sunspot equilibrium is a rational-expectations equilibrium in which the forecast is validated by actual price behavior. Before we proceed to define stationary sunspot equilibrium (SSE), we collect some useful properties of the savings function z, the rational expectations counterpart of the perfect-foresight savings function s.

Lemma 1. *The function*

$$z(R, \pi) = \arg\max_{0 \leqq y \leqq e_1} [\pi u(e_2 + y, y) + (1 - \pi) u(e_2 + Ry, y)]$$

is single-valued, continuous and such that $z(R, 0) = s(R)$ for all R, and $z(1, \pi) = s(1)$ for all π. Furthermore, $z(R, \pi)$ lies between $s(R)$ and $s(1)$ for all R and π; also $z(R, \hat{\pi})$ lies between $z(R, 0)$ and $z(R, \pi)$ if $\hat{\pi} < \pi$.

The single-valuedness and continuity of z comes from the strict concavity and continuity of the consumer's maximand w.r.t. y. The remainder of this lemma follows once we write down the first-order conditions and differentiate w.r.t. π. The key result here is that z is a simple deformation of s, with which it coincides when $\pi = 0$.

Lemma 2. *Let $\eta(R, \pi)$ be the wage elasticity of savings under stochastic beliefs w.r.t. the real wage R, evaluated at (R, π). Then $\eta(1, \pi) = (1-\pi)\varepsilon(1)$ for all π, where $\varepsilon(R)$ is the corresponding elasticity of savings under perfect foresight.*

To prove this statement, one derives an expression for $\varepsilon(1)$ by differentiating w.r.t. R the first-order conditions for the two consumer problems (perfect foresight, rational expectations), evaluating the resulting two expressions at $R = 1$, and comparing.

Having defined the function z, we are now in a position to define formally stationary sunspot equilibria.

Definition. A stationary sunspot equilibrium (SSE) is a quadruple $(p_a, p_b, \pi_{aa}, \pi_{bb})$ of positive numbers such that π_{aa} and π_{bb} lie in the open interval $(0, 1)$; $p_a \neq p_b$; and the excess demand for the consumption good is zero for each current state, i.e.

$$D^a \equiv 1/p_a - z(p_a/p_b, \pi_{aa}) = 0 \tag{4a}$$

$$D^b \equiv 1/p_b - z(p_b/p_a, \pi_{bb}) = 0. \tag{4b}$$

As a matter of convenient terminology, we say that a SSE $(p_a, p_b, \pi_{aa}, \pi_{bb})$ is a *SSE with respect to a given* (exogenous) *matrix* Π if the numbers π_{aa}, π_{bb} in our definition are diagonal elements of the matrix Π. This definition accords with the informal one proposed earlier. If event a (respectively b) occurs in the present period, p_a (respectively p_b) is actually the equilibrium price by equation (4a) and (4b). The beliefs $p_a = \phi(a)$, $p_b = \phi(b)$ are then self-fulfilling.

Note also that the definition requires both $p_a \neq p_b$ and $0 < \pi_{aa} < 1$, $0 < \pi_{bb} < 1$. If $p_a = p_b$, a SSE degenerates to a stationary equilibrium of the golden-rule type (See Lemma 1).[1] Another type of degeneracy obtains when certain transitions are ruled out in the matrix Π. In particular when $\pi_{aa} = 0$, $\pi_{bb} = 0$, the occurrence of event a (respectively b) today ensures the occurrence of b (respectively a) tomorrow. In other words, the equilibrium prices p_a and p_b necessarily succeed each other. A SSE then degenerates into a 2-cycle, as can be seen formally from Lemma 1 and equations (4a) and (4b): *two-cycles thus appear as limiting sunspot equilibria associated with a 2×2 degenerate matrix Π that has zeros in the diagonal.*

As a direct consequence of this limiting argument we have the following result.

Theorem 1 (Sunspot equilibria in the neighbourhood of two-cycles). *In an economy that admits a periodic equilibrium of order two, there is generically a neighbourhood $\nu(\bar{\Pi})$ of the 2×2 matrix Π such that a SSE exists w.r.t. every Π in $\nu(\bar{\Pi})$.*

A proof of this statement is left to the reader. It is, however, fairly intuitive and follows immediately from standard transversality theorems if the economy is identified with a sufficiently differentiable savings function. We shall reexamine the connection between sunspots and cycles more precisely in the sequel. For the time being we call *regular*[2] a periodic equilibrium of order 2 if it satisfies the conclusions of Theorem 1.

IV. AN EXISTENCE THEOREM FOR SSE

To investigate the existence of stationary sunspot equilibria, we put $w = p_a/p_b$ and define the following single-valued function

$$F(w, \pi_{aa}, \pi_{bb}) = wz(w, \pi_{aa}) - z(1/w, \pi_{bb}). \tag{5}$$

A SSE exists if, ond only if, F has a positive root $w \neq 1$ for some $\pi_{aa} \in (0, 1)$ and $\pi_{bb} \in (0, 1)$. This is so because any SSE satisfying (4a) and (4b) for some $p_a \neq p_b$, also satisfies $1/w = z(w, \pi_{aa})/z(1/w, \pi_{bb})$, and therefore $F(\cdot) = 0$. Moreover, for any positive root $w \neq 1$ of F, we can find two positive numbers, p_a and p_b, such that (4a) and (4b) hold true.

Useful properties of the function F are collected in Lemma 3.

Lemma 3. *The function $F(w, \pi_{aa}, \pi_{bb})$ is continuous for every (w, π_{aa}, π_{bb}) with $w > 0$. For each (π_{aa}, π_{bb}) it has the following properties:*

(i) $F(1, \pi_{aa}, \pi_{bb}) = 0$
(ii) $F \to \infty$ *as* $w \to \infty$.
(iii) *For w small enough,* $F(w, \pi_{aa}, \pi_{bb}) < 0$
(iv) *If w is a root of $F(w, \pi_{aa}, \pi_{bb})$, then $1/\hat{w}$ is a root of $F(w, \pi_{bb}, \pi_{aa})$.*

Proof. Parts (i) and (iv) are straightforward. Part (ii) derives from the fact that $z(w, \pi)$ is between $s(1)$ and $s(w)$ (see Lemma 1), and from standard boundary assumptions on individual behaviour, i.e. $ws(w) \to +\infty$ as $w \to \infty$. To prove (iii), we rewrite F as $w[z(w, \pi_{aa}) - (1/w)z(1/w, \pi_{bb})]$, and we note that, because of (ii), the term in brackets tends to $-\infty$ as $w \to \infty$, so that F becomes negative for small w. ||

With the assistance of Lemma 3 we may now attempt to answer two related questions. First, what can we say about the set of 2×2 transition probability matrices for which a SSE exists? Second, can we find sunspot equilibria in the neighbourhood of stationary perfect-foresight equilibria?

We begin by evaluating at $w = 1$ the derivative of the function F w.r.t. w. From Lemma 1 we obtain

$$\partial_w F(1, \pi_{aa}, \pi_{bb}) = s(1)[1 + \eta(1, \pi_{aa}) + \eta(1, \pi_{bb})]. \tag{6}$$

This combines with Lemma 2 to yield

$$\partial_w F(1, \pi_{aa}, \pi_{bb}) < 0 \quad \text{if } (2 - \pi_{aa} - \pi_{bb})\varepsilon(1) < -1. \tag{7}$$

A direct implication of (7) is

Theorem 2 (Sufficient conditions for the existence of a two-state SSE)[3]: *Suppose that the utility function satisfies regularity assumptions on differentiability, concavity and boundary behaviour. Then a sufficient condition for the existence of a sunspot equilibrium with respect to a given Markovian transition probability matrix Π is*

$$\varepsilon(1) < 0, \qquad \pi_{aa} + \pi_{bb} < 2 - \frac{1}{|\varepsilon(1)|}. \tag{8}$$

Proof. Note first that $\varepsilon(1) \geqq 0$ violates (8), and also that $F(1, \pi_{aa}, \pi_{bb}) = 0$. Given the boundary properties of F established in Lemma 3, the inequality $\partial_w F < 0$ is sufficient to ensure that F has at least one pair of roots other than $w = 1$. ||

Since this proof nowhere assumes that $\pi_{aa} > 0$, $\pi_{bb} > 0$, it applies directly to the existence of 2-cycles. The outcome is stated as Corollary 1 which merely restates Proposition 2.

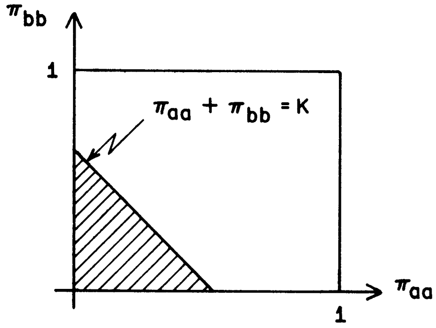

FIGURE 2

Corollary 1. *If $\varepsilon(1) < -\frac{1}{2}$, then there exists a periodic equilibrium of order* 2.

The identity of Corollary 1 and Proposition 2 is based on the fact that $\varepsilon(1) < -\frac{1}{2}$ is equivalent to local dynamic stability of the monetary stationary equilibrium.

Theorem 2 identifies a subset of the set of all two-state transition probability matrices for which there exist SSE. This subset is marked by the shaded area in Figure 2, where $K = 2 - 1/|\varepsilon(1)|$. The whole set of those matrices coincides with the unit square, and the origin represents the degenerate matrix associated with periodic two-cycles.

One thing that Theorem 2 *does not* assert is that the existence of SSE somehow depends on the $\pi_{aa} + \pi_{bb}$. As we shall see later, such a claim can only be made strictly on the borderline of the subset shaded in Figure 2, and approximately in the neighbourhood of that borderline.

V. SSE IN THE NEIGHBOURHOOD OF THE DETERMINISTIC STATIONARY EQUILIBRIUM

To grasp some of the implications of Theorem 2, we look at sunspot equilibria in the neighbourhood of the line $\pi_{aa} + \pi_{bb} = K$ in Figure 2. When $\pi_{aa} + \pi_{bb}$ decreases passing through K, $\partial_w F$ is first strictly positive, then vanishes and becomes strictly negative. The passage through zero of the derivative of F implies for that function what mathematicians call a bifurcation. We characterize that bifurcation in

Theorem 3. *Consider a one-dimensional path P on the (π_{aa}, π_{bb}) plane crossing transversally the line $\pi_{aa} + \pi_{bb} = K$ at some point C. Then the graph of $w = p_a/p_b$ as a function of the curvilinear abscissa along p has the shape of a "pitchfork" bifurcation, with only one equilibrium before point C and three equilibria after.*

The proof begins by noticing that the qualitative features of the graph will be the same along a "transversal" path and along a diagonal path with $\pi_{aa} = \pi_{bb}$. Along this latter path, because of the symmetry property of F in Lemma 3, the function F bifurcates as suggested by Figure 3(a). An elementary proof would show that two zeros arbitrarily close to 1 necessarily exist on one side of the bifurcation, and would use the implicit function theorem at these zeros. It is simpler to note that Thom's classification theorem applies here (since all vector fields on $\mathbb{R}$ are gradient vector fields): given the parameter space and symmetry properties, we have a cusp catastrophe and the section of the cusp manifold is of the pitchfork type because 1 is invariant here.

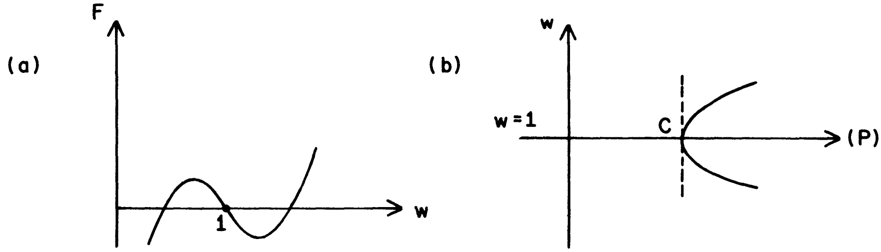

FIGURE 3

An alternative method, suggested in Woodford's survey of overlapping generations (1984), employs an approximation argument to arrive at sufficient conditions like Theorem 2. We propose below an approximation that is less formal and, we hope, more intuitive than Woodford's.

Using equation (1) we expand the excess demand function of perfect foresight about the stationary price level p^*. Putting $x_t = p_t - p^*$ one obtains

$$\varepsilon(1)x_{t+1} - [1+\varepsilon(1)]x_t = 0. \tag{9}$$

Local dynamic stability here corresponds, as we already know, to $|1+1/\varepsilon(1)| < 1$ or, equivalently, to $\varepsilon(1) < -\frac{1}{2}$.

In the rational-expectations case we define $x_a = p_a - p^*$, $x_b = p_b - p^*$ where (p_a, p_b) are sunspot prices. Then we use Lemma 2 to linearize the system consisting of equations (4a) and (4b) to:

$$(1+\pi_{ab}\varepsilon(1))x_a - \pi_{ab}\varepsilon(1)x_b = 0 \tag{10a}$$

$$\pi_{ba}x_a - (1+\pi_{ba}\varepsilon(1))x_b = 0. \tag{10b}$$

Intuitively, sunspot equilibria close to the stationary deterministic equilibrium obtain when this system has non-zero solutions in (x_a, x_b), i.e., if

$$\pi_{ab}\varepsilon(1)/[1+\pi_{ab}\varepsilon(1)] = [1+\pi_{ba}\varepsilon(1)]/\pi_{ba}\varepsilon(1) \tag{11}$$

or, equivalently, if

$$\pi_{ab} + \pi_{ba} = -1/\varepsilon(1). \tag{12}$$

We note again that $\pi_{ab} + \pi_{ba} = 2$ (equivalently, $\pi_{aa} + \pi_{bb} = 0$) requires that $\varepsilon(1) = -\frac{1}{2}$ if equation (12) is to hold. Degenerate transition probability matrices of the form $\pi_{aa} = \pi_{bb} = 0$ admit sunspot equilibria in the neighbourhood of the deterministic stationary date if that stationary state possesses borderline dynamic stability.

VI. SUNSPOT VS. PERIODIC EQUILIBRIA

The central result of this section is

Theorem 4. *Given standard assumptions on preferences and strict normal goods, a two-state stationary sunspot equilibrium exists if, and only if, a regular deterministic periodic equilibrium of order two exists.*

Proof. A regular deterministic equilibrium is by definition one for which Theorem 1 holds. Hence, the "if" part of Theorem 4 is tautologically true. The reciprocal is not obvious.

Assume accordingly that there exists a sunspot equilibrium, i.e. three positive numbers $(\bar{w}, \pi_{aa}, \pi_{bb})$ such that $\pi_{aa} < 1$, $\pi_{bb} < 1$ and $F(\bar{w}) = \bar{w}z(\bar{w}, \pi_{aa}) - z(1/\bar{w}, \pi_{bb}) = 0$. We shall demonstrate that a deterministic 2-cycle necessarily exists. To that end, we define the sets $\Omega_1 = [w \mid s(w) \geqq s(1)]$, $\Omega_2 = [w \mid s(w) \leqq s(1)]$ and prove successively the following four statements:

(S1) $\psi(w) \equiv wz(w, \pi_{aa}) < s(1)$ for $w < 1$.

(S2) There is no $w > 1$ such that $w \in \Omega_1$ and $F(w) = 0$.

(S3) There is no $w > 1$ such that $1/w \in \Omega_2$ and $F(w) = 0$.

(S4) If $F(\bar{w}) = 0$ for some $\bar{w} > 1$, than $\bar{w}s(\bar{w}) - s(1/\bar{w}) \leqq 0$.

(S1) For the first statement, we note from Lemma 1 that $s(w) \geqq z(w, \cdot) \geqq s(1)$ in Ω_1, and $s(w) \leqq z(w, \cdot) \leqq s(1)$ in Ω_2. Therefore, if $w \in \Omega_2$, then $z(w, \cdot) \leqq s(1)$ and $\psi(w) = wz(w, \cdot) \leqq ws(1) < s(1)$ for any $w < 1$. If $w \in \Omega_1$, on the other hand, then $s(1) \leqq z(w, \cdot) \leqq s(w)$ by Lemma 1 and the definition of Ω_1; therefore $\psi(w) \leqq ws(w)$. However, our normality assumption implies that $ws(w)$ is an increasing function of w, i.e. $ws(w) < s(1)$ for $w < 1$. Hence $\psi(w) < s(1)$ for $w \in \Omega_1$ s.t. $w < 1$. This completes the proof of (S1).

(S2) $F(w) = 0$ implies $z(w, \pi_{aa}) = (1/w)z(1/w, \pi_{bb})$. For $w > 1$, the right-hand side of this equality is smaller than $s(1)$, which implies $z(w, \pi_{aa}) < s(1)$ for $w < 1$, or $s(w) < s(1)$. By definition, this cannot happen for any $w \in \Omega_1$, and the proof of (S2) is complete.

(S3) This is shown in the same manner as (S2).

(S4) From (S2) we have that $\bar{w} \in \Omega_2$ and $z(\bar{w}, \cdot) > s(\bar{w})$; hence $\bar{w}z(\bar{w}, \cdot) > w\bar{s}(w)$. From (S3), on the other hand, it follows that $1/\bar{w} \in \Omega_1$, so that $z(1/\bar{w}, \cdot) \leqq s(1/\bar{w}, \cdot) \leqq s(1/\bar{w})$ and $-z(1/\bar{w}, \cdot) > -s(1/\bar{w})$. Hence, (S2) and (S3) together yield $F(\bar{w}) > \bar{w}s(\bar{w}) - s(1/\bar{w})$, which completes the proof of (S4).

Having proved the preliminary statements, we make two additional observations. First, from the symmetry of the roots of F noted previously, we assume without loss of generality that the root $\bar{w}$ of $F(w)$ *exceeds unity* (possibly after inverting a and b). Second, the function $ws(w) - s(1/w)$ becomes positive as $w \to +\infty$.

Therefore, the continuous function $ws(w) - s(1/w)$ is non-positive at $\bar{w} > 1$, and becomes strictly positive as $w \to +\infty$; it will have at least one finite real root greater than unity. ||

The reader should notice that the proof only uses the fact that z is "between" $s(w)$ and $s(1)$ in the sense of Lemma 1. A shorter proof obtains if we rely on the fact that $\eta(w, \pi)$, the wage-elasticity of z, exceeds minus one (see footnote 4). It follows from this property that $wz(w, \cdot)$ is increasing in w, and points 2 and 3 follow more immediately. However, as the above proof shows, this specific property of z is not actually needed. The main proof is more open to generalization. In fact, careful inspection of that proof suggests that Theorem 4 can be strengthened as follows:[5]

Theorem 4′. *Suppose preferences satisfy standard assumptions and strict normality. If a SSE exists relative to the matrix with diagonal elements* (π_{aa}, π_{bb}), *then a SSE also exists for every matrix with diagonal elements* $(\hat{\pi}_{aa}, \hat{\pi}_{bb})$ *such that* $\hat{\pi}_{aa} < \pi_{aa}$ *and* $\hat{\pi}_{bb} < \pi_{bb}$.

Proof. As in Theorem 4, except that (S4) must now be replaced by (S4′): $F(\bar{w}, \pi_{aa}, \pi_{bb}) = 0$ for some $\bar{w} > 1$ implies that $F(\bar{w}, \hat{\pi}_{aa}, \hat{\pi}_{bb}) < 0$ for $\hat{\pi}_{aa} < \pi_{aa}$ and $\hat{p}_{bb} < \pi_{bb}$. The rest of the argument still applies because of Lemma 2. ||

According to this result, the set of matrices associated with SSE is connected and its frontier, although not of necessity a straight line as in Figure 2, does slope downward. Behind Theorems 4 and 4′ lies the same intuition that explains two-cycles in Figure 1(b).

A two-cycle requires that, for some wage rate, the income effect of a wage change should outweigh the substitution effect by a sufficient margin. Transition probability matrices with infinitesimal diagonal elements do not alter by much the relative strength of substitution and income effects. Therefore, the continuity of savings behaviour implies that a SSE exists whenever a two-cycle does.

Suppose, on the other hand, that a SSE exists for some transition probability matrix Π and that $\varepsilon(1) < 0$. Then Lemma 2 says that, if we reduce the size of diagonal elements in Π, we strengthen the income effect of a wage change relative to the substitution effect and facilitate the existence of stationary sunspot equilibria. An extreme point of this process is when the diagonal elements vanish altogether and we obtain a deterministic two-cycle.

VII. CONCLUSIONS

We review here the main results from earlier sections and discuss the prospects for generalizing each of them. The summary relies heavily on the diagrams of Figure 4; each panel in that figure graphs the SSE price ratio w against a one dimensional parameter—say, $\pi = \pi_{aa} + \pi_{bb}$ for simplicity—that stands for the transition probability matrix Π.

Theorem 1, depicted in panel (a), says that the two broken-line branches exist for any 2-cycle. This theorem is a formal elaboration of a simple idea which, to the best of our knowledge, does not appear in the literature antedating the present paper. The

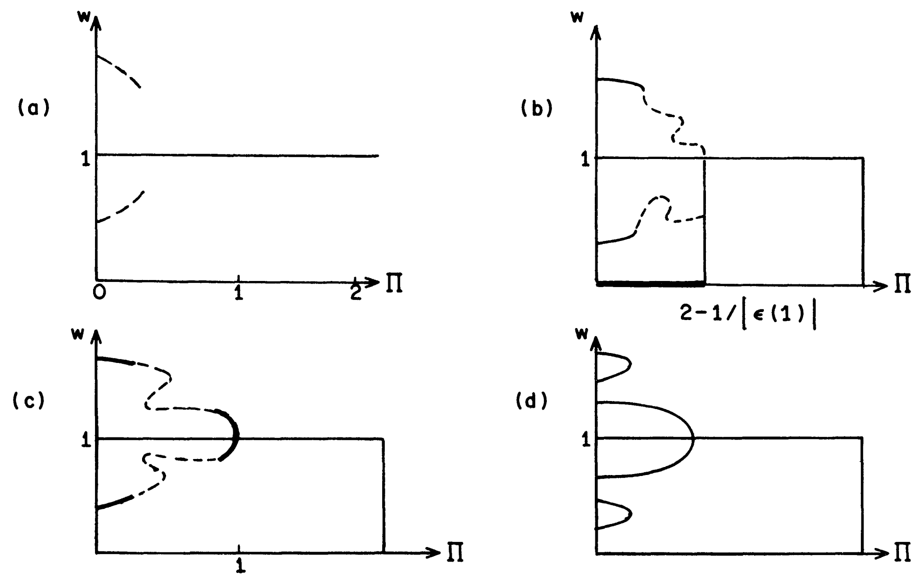

FIGURE 4

existence of sunspot equilibrium in the neighbourhood of periodic cycles is potentially a very general property, holding for n-dimensional systems (i.e. for n physical commodities) in which expectations look one period forward and for cycles of any order k.

Theorem 2, on the other hand, means that the graph is non-empty above the thick line of panel (b). The principle of the method used for proving Theorem 2 is quite general. The argument based on the slope of the tangent is, in fact, a one-dimensional version of the Poincaré–Hopf theorem. A generalization of that sufficient condition to n-dimensional systems is obtained in a forthcoming paper of Guesnerie (1985).

The same methodology applies in generalizing theorems 2 and 3 to cycles of order k, either in one-or multi-dimensional systems. We note, meantime, that the existence of sunspots of order k follows generically from the existence of sunspots of order 2.[6] Let us sketch here the argument for passing from $k=2$ to $k=3$. Consider a SSE of order 2 associated with two events a and b. Then add to these a third event c such that the probability of passage from a to c is zero, the one from b to c is also zero, and the probability of passing from c to c is unity. If (p_a, p_b) are equilibrium prices in a SSE of order 2, and p^* is the equilibrium price at the stationary monetary equilibrium, then clearly the vector (p_a, p_b, p_c) is a "degenerate" SSE of order 3 if $p_c = p^*$. Perturbing slightly the transition probability matrix will yield a regular SSE of order 3.

Panel (c) demonstrates the bifurcation studied in Theorem 3. The pitchfork aspect of that bifurcation seems to be a characteristic of the bifurcation for sunspots of order 2, even in more general systems. However, sunspots of higher order have necessarily more complex bifurcations. Finally, panel (d) provides one possible illustration of the details that Theorems 4 and 4′ add to our knowledge of sunspot equilibria. We note, nevertheless, that the one-dimensional parametrization of transition probability matrices wastes some information on the set of matrices compatible with stationary sunspot equilibria.

APPENDIX: TWO EXAMPLES OF DYNAMIC SUNSPOT EQUILIBRIUM

Example 1 (Sunspot equilibrium is a lottery on multiple temporary equilibria). Suppose the offer curve bends backward, i.e. the forward-looking solution to $D(p_t, p_{t+1}) = 0$ is not unique. Then there exist equilibrium price sequences $(p_1, p_2, \ldots)$ and $(p_1, p_2^1, \ldots)$ such that $p_2 \neq p_2^1$. Furthermore, it must be the case that $y = 1/p_1$ maximizes both $u(p_1 y/p_2 + e_2, y)$ and $u(p_1 y/p_2^1 + e_2, y)$ over the interval $[0, e_1]$. It follows that the same value of y maximizes over the same interval the function $\pi u(p_1 y/p_2 + e_2, y) + u(p_1 y/p_2^1 + e_2, y)$ for any arbitrary $\pi \in [0, 1]$. Therefore, a third equilibrium price sequence $(p_1, \tilde{p}_2, \ldots)$ exists, where $\tilde{p}_2$ is a random variable with realizations $\tilde{p}_2 = p_2$ w.p. π; $= p_2^1$ w.p. $1 - \pi$.

Example 2 (Multiple temporary equilibria are not necessary for the existence of dynamical sunspot equilibria). Suppose now that endowments are $e_1 > 0$, $e_2 = 1$, and preferences are given by $u = c_{t+1} - (1/2)y_t^2$. Then, in perfect foresight, the savings function equals p_t/p_{t+1}, and excess demand is $D(p_t, p_{t+1}) = 1/p_t - p_t/p_{t+1}$. An equilibrium price sequence (p_t) exists for each $p_t \geqq 1$, and it is unique; at $p_t = 1$ we support the golden-rule allocation.

With extraneous uncertainty, individuals maximize expected utility conditional on price and other observations. Specifically, saving now equals $p_t E(p_t^{-1} \mid I_t)$, where I_t is the conditioning information for t, and excess demand is

$$d(p_t, I_t) = 1/p_t - p_t E(p_{t+1}^{-1} \mid I_t). \tag{13}$$

We introduce now two arbitrary numbers $\theta \in (0, 1)$ and $q_1 > 1$, and define two infinite scalar sequences $(q_t)_{t=2}^{\infty}$ and $(\pi_t)_{t=2}^{\infty}$ recursively from

$$q_{t+1} = q_t^2/\theta, \qquad \pi_{t+1} = \frac{1-\theta}{q_t^2 - \theta}. \tag{14}$$

Since $q_{t+1} > q_t$ and $q_1 > 1$, we have $q_t > 1$ for $t = 1, 2, \dots$, and $\pi_t \in (0, 1)$ for $t = 2, \dots$.

One verifies easily that the sequence $(\tilde{p}_t)_{t=2}^{\infty}$ of random variables with realizations

$$p_t = p_{t-1} \quad \text{if } p_{t-1} = 1$$

$$p_t = \begin{cases} q_t & \text{w.p.} \quad 1-\pi_t \\ 1 & \text{w.p.} \quad \pi_t \end{cases}, \quad \text{if } p_{t-1} \neq 1 \tag{15}$$

constitutes a sunspot equilibrium, that is, satisfies $d(p_t, I_t) = 0$. By construction, the sequence $(\tilde{p}_t)$ converges to the golden rule with some probability $Q > 0$, and to autarky with probability $1 - Q$.

First version received November 1984; *final version accepted March* 1986 (*Eds*)

An earlier version of this paper circulated in February 1982 under the title "The Persistence of Self-Fulfilling Theories". We are indebted to the National Science Foundation for financial support, to the Ecole des Hautes Etudes en Sciences Sociales for hospitality, to Jean-Michel Grandmont, Jean Tirole and Michael Woodford for enlightening discussions, and to Jim Peck for discovering an error in Example 2. We are also grateful for the helpful suggestions of two anonymous referees. All remaining errors are ours.

NOTES

1. The golden rule is typically defined in connection with stationary consumption optima in growth problems: at $m = m^*$, stationary ordinal utility $u(e_2 + m, m)$ attains a maximum. See Phelps (1961).
2. Note that this definition is not standard. Regular equilibria in the usual sense are regular in our sense but the converse is not necessarily true.
3. Another equivalent sufficient condition of SSE equilibria has been given in Azariadis-Guesnerie (1982), where condition (8) is also stated without complete proof. See Spear (1984) for a similar result.
4. It is easy to conclude in fact that $\eta(w, \pi) > \eta(w, 0)$, which itself is greater than -1.
5. We are grateful to an anonymous referee for suggesting this extension.
6. This would not be true of deterministic cycles. It is therefore possible, as Woodford (1984) points out, for higher-order SSE to exist even though the corresponding periodic cycle does not.

REFERENCES

AZARIADIS, C. (1981), "Self-Fulfilling Prophecies", *Journal of Economic Theory*, **25**, 380-396.

AZARIADIS, C. and GUESNERIE, R., (1982), "Prophéties Créatices et Persistence des Théories", *Review Economique*, **33**, 787-806.

CASS, D., OKUNO, M. and ZILCHA, I. (1979), "The Role of Money in Supporting the Pareto Optimality of Competitive Equilibrium in Consumption-Loan Type Models", *Journal of Economic Theory*, **20**, 41-80.

CASS, D. and SHELL, K., (1983), "Do Sunspots Matter?", *Journal of Political Economy*, **91**, 193-227.

GALE, D. (1973), "Pure Exchange Equilibrium of Dynamic Economic Models", *Journal of Economic Theory*, **6**, 12-36.

GOURIÉROUX, C., LAFFONT, J.-J. and MONTFORT, A. (1982), "Rational Expectations in Dynamical Linear Models: Analysis of the Solutions", *Econometrica*, **50**, 409-426.

GRANDMONT, J.-M. (1985), "On Endogenous Competitive Business Cycles", *Econometrica*, **53**, 995-1045.

GROSSMAN, S. and STIGLITZ, J. (1976), "Information and Competitive Price Systems", *American Economic Review*, **66**, 246-253.

GUESNERIE, R. (1985), "Stationary Sunspot Equilibria in an *n*-Commodity World", (mimeo, EHESS, Paris).

LUCAS, R. (1972), "Expectations and the Neutrality of Money", *Journal of Economic Theory*, **4**, 101-124.

MASKIN, E. and TIROLE, J. (1985), "Correlated Equilibria and Sunspots: A Note" (mimeo, Harvard University).

PECK, J. (1984), "On the Existence of Sunspot Equilibria in an Overlapping Generation Model" (mimeo, University of Pennsylvania).

PHELPS, E. (1961), "The Golden Rule of Accumulation: A Fable for Growthmen", *American Economic Review*, **51**, 638-643.

RADNER, R. (1979), "Rational Expectations Equilibrium: Generic Existence and the Information Revealed by Prices", *Econometrica*, **47**, 655-678.

SAMULESON, P. (1958), "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money", *Journal of Political Economy*, **66**, 467-482.

SHELL, K. (1977), "Monnaie et Allocation Intertemporelle" (CEPREMAP memorandum).

SHILLER, R. (1978), "Rational Expectations and the Dynamical Structure of Macroeconomic Models", *Econometrica*, **46**, 467-482.

SPEAR, S. (1984), "Sufficient Conditions for the Existence of Sunspot Equilibria", *Journal of Economic Theory*, **34**, 360-370.

TAYLOR, J. (1977), "Conditions for Unique Solutions in Stochastic Macroeconomic Models", *Econometrica*, **45**, 1377-1385.

WOODFORD, M. (1984), "Indeterminacy of Equilibrium in the Overlapping Generations Model: A Survey" (mimeo, University of Pennsylvania).